

Algebras of invariant differential operators on unit sphere bundles over two-point homogeneous Riemannian spaces

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2003 J. Phys. A: Math. Gen. 36 7361

(<http://iopscience.iop.org/0305-4470/36/26/310>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.86

The article was downloaded on 02/06/2010 at 16:20

Please note that [terms and conditions apply](#).

Algebras of invariant differential operators on unit sphere bundles over two-point homogeneous Riemannian spaces

A V Shchepetilov

Department of Physics, Moscow State University, 119992 Moscow, Russia

E-mail: alexey@quant.phys.msu.su

Received 11 February 2003, in final form 9 May 2003

Published 18 June 2003

Online at stacks.iop.org/JPhysA/36/7361

Abstract

Let G be the identity component of the isometry group for an arbitrary curved two-point homogeneous space M . We consider algebras of G -invariant differential operators on bundles of unit spheres over M . The generators of this algebra and the corresponding relations for them are found. The connection of these generators with the two-body problem on two-point homogeneous spaces is discussed.

PACS numbers: 02.40.Vh, 02.40.Ky, 02.20.Qs, 03.65.Fd

Mathematics Subject Classification: 16S32, 43A85, 22F30, 70F05

1. Introduction

The property of a differential operator on a smooth manifold M to be invariant with respect to an action of some group G (especially a Lie group) on M plays a great role in mathematical physics since it helps select physically significant operators. The algebra $\text{Diff}(M)$ of all G -invariant differential operators with complex or real coefficients on M gives the material for constructing G -invariant physical theories on M . Properties of such a theory are in close connection with properties of the algebra $\text{Diff}(M)$.

A homogeneous smooth manifold M of the Lie group G is called *commutative space*, if the algebra $\text{Diff}(M)$ is commutative. The well-known example of a commutative space is the symmetric space of rank l . Recall that the *rank of a symmetric space* is the dimension of its maximal flat completely geodesic submanifold. The commutative algebra $\text{Diff}(M)$ for this space is generated by l independent commutative generators [1]. Particularly, for symmetric spaces of rank 1 (which are the same as two-point homogeneous spaces) the algebra $\text{Diff}(M)$ is generated by the Laplace–Beltrami operator. Also, the class of commutative spaces contains weakly symmetric spaces [2].

Only some sporadic examples of noncommutative algebras $\text{Diff}(M)$ are known (see, for example, ch 2 [3]). One of these examples is the noncommutative algebra $\text{Diff}(M_1)$ for

$M_1 = O_0(1, n)/SO(n - 1)$ studied in [4], where $O_0(1, n)$ is the identity component for the group $O(1, n)$. In that paper the space M_1 was interpreted as the total space for the bundles of unit spheres over the hyperbolic space $\mathbf{H}^n(\mathbb{R})$. Denote the total space of the bundle of unit spheres over a Riemannian space M by $M_{\mathbb{S}}$.

The space $\mathbf{H}^n(\mathbb{R})$ is a representative of the class of two-point homogeneous Riemannian spaces (TPHRS) for which any pair of points that can be transformed by means of appropriate isometry to any other pair of points with the same distance between them. Equivalently, these spaces are characterized by the property that the natural action of the isometry group on the bundle of unit spheres over them is transitive. Thus the natural problem arises: *describe the algebras $\text{Diff}(M_{\mathbb{S}})$ for the bundle of unit spheres over all TPHRS M .*

From the point of view of the two-body problem in classical and quantum mechanics, TPHRS are characterized by the property that the distance between particles is the only invariant of the isometry group G in the configuration space. The space $M_{\mathbb{S}}$ is isomorphic to an orbit in general position for the symmetry group G of the two-body problem on the TPHRS M acting in the configuration space of this problem. Due to the two-point homogeneity of M the codimension of these orbits is one. Thus, for the two-body problem in TPHRS there is the degree of freedom corresponding to the distance between particles; other degrees of freedom correspond to the homogeneous manifold and can be described in terms of the symmetry group G .

In [5] the polynomial expression for the Hamiltonian \hat{H} of the quantum mechanical two-body problem on an arbitrary TPHRS M was found through the radial differential operator and elements of $\text{Diff}(M_{\mathbb{S}})$. In the present paper the generators of algebras $\text{Diff}(M_{\mathbb{S}})$ and the corresponding relations for them are found for all curved two-point homogeneous spaces M . Some properties of these generators are discussed.

This paper is organized as follows. In section 2, we give the necessary information on invariant differential operators on homogeneous spaces. We recall the classification of TPHRS in section 3. In section 4, we specify the construction of invariant differential operators for the space $M_{\mathbb{S}}$, where M is a two-point homogeneous Riemannian space. Some generators are found common for all $\text{Diff}(M_{\mathbb{S}})$. In section 5, the model of the quaternion projective space $\mathbf{P}^n(\mathbb{H})$ is described. In section 6, the generators for the algebra $\text{Diff}(\mathbf{P}^n(\mathbb{H})_{\mathbb{S}})$ are calculated and by formal correspondence the analogous generators of the algebra $\text{Diff}(\mathbf{H}^n(\mathbb{H})_{\mathbb{S}})$ are obtained. The corresponding relations for these algebras are found in section 7. In section 8 we consider from the same point of view algebras $\text{Diff}(\mathbf{P}^n(\mathbb{C})_{\mathbb{S}})$, $\text{Diff}(\mathbf{H}^n(\mathbb{C})_{\mathbb{S}})$ and in section 9 the algebras $\text{Diff}(\mathbf{P}^n(\mathbb{R})_{\mathbb{S}})$, $\text{Diff}(\mathbf{S}_{\mathbb{S}}^n)$. In section 10, there is a description of the Cayley plane $\mathbf{P}^2(\mathbb{C}a)$ through the exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{C}a)$. In section 11, the generators for the algebra $\text{Diff}(\mathbf{P}^2(\mathbb{C}a)_{\mathbb{S}})$ are calculated and by formal correspondence the analogous generators of the algebra $\text{Diff}(\mathbf{H}^2(\mathbb{C}a)_{\mathbb{S}})$ are obtained. The corresponding relations for these algebras are found in section 12.

The connection of constructed generators with the two-body problem on two-point homogeneous spaces is discussed in section 13.

In appendix A we describe the technique for calculating the commutative relations for algebras of differential operators under consideration and in appendix B one interesting fact for an arbitrary TPHRS is proved.

2. Invariant differential operators on homogeneous spaces

Let G be a Lie group, M be a Riemannian G -homogeneous left space, $x_0 \in M$, $K \subset G$ be the stationary subgroup of the point $x_0 \in M$ and $\mathfrak{k} \subset \mathfrak{g} \equiv T_e G$ be the corresponding Lie algebras.

Choose a subspace $\mathfrak{p} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ (a direct sum of linear spaces). The space \mathfrak{p} can be identified with the tangent space $T_{x_0}M$.

The stationary subgroup K is compact, since it is also the subgroup of the group $SO(n)$. By averaging on the group K we can define a Ad_K -invariant scalar product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} and choose the subspace \mathfrak{p} orthogonal to \mathfrak{k} with respect to this product [3, 6]. In this case we have the inclusion $\text{Ad}_K(\mathfrak{p}) \subset \mathfrak{p}$, i.e. the space M is reductive.

Identify the space M with the factor space of left conjugate classes of the group G with respect to the subgroup K . Let $\pi : G \rightarrow G/K$ be the natural projection.

Let $S(V)$ be a graded symmetric algebra over a finite-dimensional complex space V , i.e. a free commutative algebra over the field \mathbb{C} , generated by elements of any basis of V . The adjoint action of the group G on \mathfrak{g} can be naturally extended to the action of G on the algebra $S(\mathfrak{g})$ according to the formula:

$$\text{Ad}_g : Y_1 \cdots Y_i \rightarrow \text{Ad}_g(Y_1) \cdots \text{Ad}_g(Y_i), Y_1, \dots, Y_i \in \mathfrak{g}.$$

Denote by \mathfrak{g}^K the set of all Ad -invariants in $S(\mathfrak{g})$.

The main result of the invariant differential operators theory is the existence of the one to one correspondence between the algebra $\text{Diff}(M)$ and the set \mathfrak{p}^K of all Ad_K -invariants in $S(\mathfrak{p})$ [3]. For our purpose the next version [2] of this result is more convenient. Let $U(\mathfrak{g})$ be the universal enveloping algebra with the standard filtration for the Lie algebra \mathfrak{g} and $U(\mathfrak{g})^K$ be its subalgebra, consisting of all Ad_K -invariant elements in $U(\mathfrak{g})$. Let μ be the linear mapping of $S(\mathfrak{p})$ into $U(\mathfrak{g})$, according to the formula

$$\mu(Y_1 \cdots Y_p) = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} Y_{\sigma(1)} \cdots Y_{\sigma(p)}$$

where on the left-hand side the element $Y_1 \cdots Y_p$ is supposed to be in $S(\mathfrak{p})$ and on the right-hand side it is supposed to be in $U(\mathfrak{g})$. Here \mathfrak{S}_p is the permutation group of p elements. Obviously, μ is injective.

Let $U(\mathfrak{g})\mathfrak{k}$ be the left ideal in $U(\mathfrak{g})$ generated by \mathfrak{k} and $(U(\mathfrak{g})\mathfrak{k})^K$ be the set of all Ad_K -invariant elements in $U(\mathfrak{g})\mathfrak{k}$. The set $(U(\mathfrak{g})\mathfrak{k})^K$ is a two-sided ideal in $U(\mathfrak{g})^K$ since for elements $f \in \mathfrak{k}$ and $g \in U(\mathfrak{g})^K$ we have $fg = \text{ad}_f g + gf = gf$. Also $\mu(\mathfrak{p}^K) \subset U(\mathfrak{g})^K$, because M is reductive. Hence we can define the factor algebra $U(\mathfrak{g})^K / (U(\mathfrak{g})\mathfrak{k})^K$. Let $\eta : U(\mathfrak{g})^K \rightarrow U(\mathfrak{g})^K / (U(\mathfrak{g})\mathfrak{k})^K$ be the canonical projection.

Theorem 1 ([2]). *The algebras $\text{Diff}(M)$ and $U(\mathfrak{g})^K / (U(\mathfrak{g})\mathfrak{k})^K$ are isomorphic.*

Every element of $U(\mathfrak{g})^K / (U(\mathfrak{g})\mathfrak{k})^K$ has the unique representative from \mathfrak{p}^K or equivalently from $\mu(\mathfrak{p}^K)$. We can get the relations in $\text{Diff}(M)$ operating in $U(\mathfrak{g})^K$ modulo $(U(\mathfrak{g})\mathfrak{k})^K$. This approach leads to simpler calculations than the operations through local coordinates on M (like in [4]) that gives quite cumbersome calculations even in the relatively simple case of $M = \mathbf{H}^n(\mathbb{R})_{\mathbb{S}}$.

Below we are interested in the representation of the associative algebra $\text{Diff}(M)$ by its generators and corresponding relations. Let $\{g_i\}$ be a set of generators of the commutative subalgebra $\mathfrak{p}^K \subset S(\mathfrak{p})$. Not losing generality we can suppose that all g_i are homogeneous elements w.r.t. the grading of $S(\mathfrak{p})$. Then the elements $\eta \circ \mu(g_i)$ generate the algebra $\text{Diff}(M)$.

Relations for the elements $\eta \circ \mu(g_i)$ are of two types. The first type consists of relations induced by relations in $U(\mathfrak{g})$. Due to the universality of $U(\mathfrak{g})$ all these relations are commutative ones, induced by the Lie operation in \mathfrak{g} . They are reduced to *commutative relations* of the simplest form: $[D_1, D_2] = \tilde{D}$, where the operators $D_1, D_2 \in \text{Diff}(M)$ have degrees m_1 and m_2 respectively and the degree of $\tilde{D} \in \text{Diff}(M)$ is less than or equal to $m_1 + m_2 - 1$.

Suppose now that there is a relation in $U(\mathfrak{g})$ of the form

$$P(\eta \circ \mu(g_1), \dots, \eta \circ \mu(g_k)) = 0$$

or equivalently

$$P(\mu(g_1), \dots, \mu(g_k)) = \tilde{D} \quad (1)$$

where P is a polynomial and $\tilde{D} \in (U(\mathfrak{g})\mathfrak{k})^K$. Using the commutative relations for $\mu(g_i)$, $i = 1, \dots, k$ we can reduce the polynomial P to the polynomial P_1 , symmetric w.r.t. all permutations of its arguments, and equation (1) becomes

$$P_1(\mu(g_1), \dots, \mu(g_k)) = D^* \quad (2)$$

$D^* \in (U(\mathfrak{g})\mathfrak{k})^K$. After this reduction relation (2) may be trivial: $P_1 = 0$, $D^* = 0$. It means that (1) is the commutative relation. Suppose that relation (2) is nontrivial. Consider the sum $P_2(t_1, \dots, t_k)$ of monomials with the highest total degree from polynomial $P_1(t_1, \dots, t_k)$ with commutative variables t_1, \dots, t_k . Due to the symmetry of P_1 the polynomial $P_2(t_1, \dots, t_k)$ is nontrivial. On the other hand from (2) we obtain that $P_2(g_1, \dots, g_k) = 0$ due to the expansion $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. Thus every relation in the algebra $U(\mathfrak{g})^K / (U(\mathfrak{g})\mathfrak{k})^K$ w.r.t. its generators $\eta \circ \mu(g_1), \dots, \eta \circ \mu(g_k)$ modulo commutative relations is in one to one correspondence with the relations for homogeneous generators g_1, \dots, g_k of the commutative algebra \mathfrak{p}^K . We call such relations *the relations of the second type*.

The filtration of the algebra $U(\mathfrak{g})$ induces the filtration of the algebra $\text{Diff}(M)$ which coincides with the natural filtration of $\text{Diff}(M)$ as the algebra of differential operators.

For simplicity throughout the whole paper we consider invariance of differential operators only w.r.t. the identity component of a whole isometry group.

3. Two-point homogeneous Riemannian spaces

The classification of two-point homogeneous Riemannian spaces can be found in [7, 8] (see also [9–11]), and is as follows:

1. the Euclidean space \mathbf{E}^n ;
2. the sphere \mathbf{S}^n ;
3. the real projective space $\mathbf{P}^n(\mathbb{R})$;
4. the complex projective space $\mathbf{P}^n(\mathbb{C})$;
5. the quaternion projective space $\mathbf{P}^n(\mathbb{H})$;
6. the Cayley projective plane $\mathbf{P}^2(\mathbb{C}a)$;
7. the real hyperbolic space (Lobachevski space) $\mathbf{H}^n(\mathbb{R})$;
8. the complex hyperbolic space $\mathbf{H}^n(\mathbb{C})$;
9. the quaternion hyperbolic space $\mathbf{H}^n(\mathbb{H})$;
10. the Cayley hyperbolic plane $\mathbf{H}^2(\mathbb{C}a)$.

The isometry groups for all these spaces except Cayley planes are classical and for the Cayley planes they are two real forms of the complex special simple group \mathbf{F}_4 .

For the Lie algebra \mathfrak{g} of the isometry group of the compact two-point homogeneous space M there is the following general expansion [5, 12, 13], which is the specification of the expansion $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ from section 2.

Proposition 1. *The algebra \mathfrak{g} admits the following expansion into the direct sum of subspaces:*

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{k}_0 \oplus \mathfrak{k}_\lambda \oplus \mathfrak{k}_{2\lambda} \oplus \mathfrak{p}_\lambda \oplus \mathfrak{p}_{2\lambda} \quad (3)$$

where $\dim \mathfrak{a} = 1$, λ is a nontrivial linear form on the space \mathfrak{a} , $\dim \mathfrak{k}_\lambda = \dim \mathfrak{p}_\lambda = q_1$, $\dim \mathfrak{k}_{2\lambda} = \dim \mathfrak{p}_{2\lambda} = q_2$, $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_\lambda \oplus \mathfrak{p}_{2\lambda}$, $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_\lambda \oplus \mathfrak{k}_{2\lambda}$; $q_1, q_2 \in \{0\} \cup \mathbb{N}$, the subalgebra

\mathfrak{a} is the maximal commutative subalgebra in the subspace \mathfrak{p} . Here \mathfrak{k} is a stationary subalgebra, corresponding to some point $x_0 \in M$. All summands in (3) are $\text{ad}_{\mathfrak{k}_0}$ -invariant; in particular \mathfrak{k}_0 is a subalgebra of \mathfrak{k} .

Identify the space \mathfrak{p} with the space $T_{x_0}M$. Under this identification the restriction of the Killing form for the algebra \mathfrak{g} onto the space \mathfrak{p} and the scalar product on $T_{x_0}M$ are proportional. In particular, the decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ is uniquely determined by the point x_0 . Let $\langle \cdot, \cdot \rangle$ be a scalar product on the algebra \mathfrak{g} such that it is proportional to the Killing form and its restriction onto the subspace $\mathfrak{p} \cong T_{x_0}M$ coincides with the Riemannian metric g on $T_{x_0}M$. The spaces \mathfrak{a} , \mathfrak{k}_0 , \mathfrak{k}_λ , \mathfrak{p}_λ , $\mathfrak{k}_{2\lambda}$, $\mathfrak{p}_{2\lambda}$ are pairwise orthogonal.

Besides, the following inclusions are valid:

$$\begin{array}{llll}
 [\mathfrak{a}, \mathfrak{p}_\lambda] \subset \mathfrak{k}_\lambda & [\mathfrak{a}, \mathfrak{k}_\lambda] \subset \mathfrak{p}_\lambda & [\mathfrak{a}, \mathfrak{p}_{2\lambda}] \subset \mathfrak{k}_{2\lambda} & [\mathfrak{a}, \mathfrak{k}_{2\lambda}] \subset \mathfrak{p}_{2\lambda} \\
 [\mathfrak{a}, \mathfrak{k}_0] = 0 & [\mathfrak{k}_\lambda, \mathfrak{p}_\lambda] \subset \mathfrak{p}_{2\lambda} \oplus \mathfrak{a} & [\mathfrak{k}_\lambda, \mathfrak{k}_\lambda] \subset \mathfrak{k}_{2\lambda} \oplus \mathfrak{k}_0 & \\
 [\mathfrak{p}_\lambda, \mathfrak{p}_\lambda] \subset \mathfrak{k}_{2\lambda} \oplus \mathfrak{k}_0 & [\mathfrak{k}_{2\lambda}, \mathfrak{k}_{2\lambda}] \subset \mathfrak{k}_0 & [\mathfrak{p}_{2\lambda}, \mathfrak{p}_{2\lambda}] \subset \mathfrak{k}_0 & [\mathfrak{k}_{2\lambda}, \mathfrak{p}_{2\lambda}] \subset \mathfrak{a} \\
 [\mathfrak{k}_\lambda, \mathfrak{k}_{2\lambda}] \subset \mathfrak{k}_\lambda & [\mathfrak{k}_\lambda, \mathfrak{p}_{2\lambda}] \subset \mathfrak{p}_\lambda & [\mathfrak{p}_\lambda, \mathfrak{k}_{2\lambda}] \subset \mathfrak{p}_\lambda & [\mathfrak{p}_\lambda, \mathfrak{p}_{2\lambda}] \subset \mathfrak{k}_\lambda.
 \end{array} \quad (4)$$

Moreover, for any basis $e_{\lambda,i}$, $i = 1, \dots, q_1$ in the space \mathfrak{p}_λ and any basis $e_{2\lambda,i}$, $i = 1, \dots, q_2$ in the space $\mathfrak{p}_{2\lambda}$ there are the basis $f_{\lambda,i}$, $i = 1, \dots, q_1$ in the space \mathfrak{k}_λ and the basis $f_{2\lambda,j}$, $j = 1, \dots, q_2$ in the space $\mathfrak{k}_{2\lambda}$ such that

$$\begin{array}{llll}
 [\Lambda, e_{\lambda,i}] = -\frac{1}{2}f_{\lambda,i} & [\Lambda, f_{\lambda,i}] = \frac{1}{2}e_{\lambda,i} & [e_{\lambda,i}, f_{\lambda,i}] = -\frac{1}{2}\Lambda & \\
 \langle e_{\lambda,i}, e_{\lambda,j} \rangle = \langle f_{\lambda,i}, f_{\lambda,j} \rangle = \delta_{ij}R^2 & i, j = 1, \dots, q_1 & & \\
 [\Lambda, e_{2\lambda,i}] = -f_{2\lambda,i} & [\Lambda, f_{2\lambda,i}] = e_{2\lambda,i} & [e_{2\lambda,i}, f_{2\lambda,i}] = -\Lambda & \\
 \langle e_{2\lambda,i}, e_{2\lambda,j} \rangle = \langle f_{2\lambda,i}, f_{2\lambda,j} \rangle = \delta_{ij}R^2 & i, j = 1, \dots, q_2 & \langle \Lambda, \Lambda \rangle = R^2 &
 \end{array} \quad (5)$$

where the vector $\Lambda \in \mathfrak{a}$ satisfies the conditions $\langle \Lambda, \Lambda \rangle = R^2$, $|\lambda(\Lambda)| = \frac{1}{2}$. Here the positive constant R is connected with the maximal sectional curvature κ_m of the space M by the formula $\kappa_m = R^{-2}$.

Nonnegative integers q_1 and q_2 are said to be *multiplicities of the space M* . They characterize M uniquely up to the exchange $\mathbf{S}^n \leftrightarrow \mathbf{P}^n(\mathbb{R})$. For the spaces \mathbf{S}^n and $\mathbf{P}^n(\mathbb{R})$ we have $q_1 = 0, q_2 = n - 1$; for the space $\mathbf{P}^n(\mathbb{C})$: $q_1 = 2n - 2, q_2 = 1$; for the space $\mathbf{P}^n(\mathbb{H})$: $q_1 = 4n - 4, q_2 = 3$ and for the space $\mathbf{P}^2(\mathbb{C}a)$: $q_1 = 8, q_2 = 7$. Conversely, for the spaces \mathbf{S}^n and $\mathbf{P}^n(\mathbb{R})$ we could reckon that $q_1 = n - 1, q_2 = 0$. Our choice corresponds to the isometries $\mathbf{P}^1(\mathbb{C}) \cong \mathbf{S}^2$, $\mathbf{P}^1(\mathbb{H}) \cong \mathbf{S}^4$.

Remark 1. Noncompact two-point homogeneous spaces of types 7, 8, 9, 10 are analogous to the compact two-point homogeneous spaces of types 2, 4, 5, 6, respectively. In particular, it means that Lie algebras \mathfrak{g} of symmetry groups of analogous spaces are different real forms of a simple complex Lie algebra. The transition from one such real form to another can be done by multiplying the subspace \mathfrak{p} from the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ by the imaginary unit \mathbf{i} (or by $-\mathbf{i}$).

The information concerning the representation theory of symmetry groups of two-point homogeneous spaces can be found in [14].

4. Invariant differential operators on $M_{\mathbb{S}}$

Here we shall specify the construction from section 2 for the space $M_{\mathbb{S}}$, where M is a two-point homogeneous compact Riemannian space, using proposition 1.

Let G be the identity component of the isometry group for M and K be its stationary subgroup, corresponding to the point $x_0 \in M$. The group G naturally acts on the space $M_{\mathbb{S}}$ and this action is transitive [10] (lemma 8.12.1). In particular K acts transitively on the unit sphere $\mathbb{S}_{x_0} \subset T_{x_0}M$. Identify the space \mathfrak{p} from proposition 1 with the space $T_{x_0}M$. After this identification the action of K on $T_{x_0}M$ becomes the adjoint action Ad_K on \mathfrak{p} . Let K_0 be the subgroup of K , corresponding to the subalgebra $\mathfrak{k}_0 \subset \mathfrak{k}$. Due to relations (4) and (5) K_0 is the stationary subgroup of the group G , corresponding to the point¹ $y := (x_0, \Lambda') \in M_{\mathbb{S}}$, where $\Lambda' := \frac{1}{R}\Lambda$.

Let $\tilde{\mathfrak{p}} := \mathfrak{a} \oplus \mathfrak{p}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_\lambda \oplus \mathfrak{k}_{2\lambda}$. Since $[\mathfrak{k}_0, \tilde{\mathfrak{p}}] \subset \tilde{\mathfrak{p}}$ the expansion $\mathfrak{g} = \tilde{\mathfrak{p}} \oplus \mathfrak{k}_0$ is reductive. One has $T_y M_{\mathbb{S}} = T_{x_0}M \oplus T_{\Lambda'}\mathbb{S}_{x_0}$. Due to proposition 1 we obtain $\Lambda' \perp (\mathfrak{p}_\lambda \oplus \mathfrak{p}_{2\lambda})$ and $[f_{\lambda,i}, \Lambda'] = -(2R)^{-1}e_{\lambda,i}, i = 1, \dots, q_1, [f_{2\lambda,j}, \Lambda'] = -R^{-1}e_{2\lambda,j}, j = 1, \dots, q_2$. Therefore, the space $\mathfrak{k}_\lambda \oplus \mathfrak{k}_{2\lambda}$ is identified through the K -action on $T_{x_0}M$ with the space $T_{\Lambda'}\mathbb{S}_{x_0}$ and the K_0 -action on the space $T_y M_{\mathbb{S}} \simeq \tilde{\mathfrak{p}} = \mathfrak{a} \oplus \mathfrak{p}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_\lambda \oplus \mathfrak{k}_{2\lambda}$ is again adjoint.

From proposition 1 we see that Ad_{K_0} conserves all summands in the last expansion. On the other hand, the K_0 -action on $T_{\Lambda'}\mathbb{S}_{x_0}$ is the differential of K_0 -action on $(\Lambda')^\perp \subset T_{x_0}M$. Since the last action is linear we obtain that the Ad_{K_0} -action on \mathfrak{p}_λ is equivalent to its action on \mathfrak{k}_λ and the Ad_{K_0} -action on $\mathfrak{p}_{2\lambda}$ is equivalent to its action on $\mathfrak{k}_{2\lambda}$. Let $\chi_\lambda : \mathfrak{k}_\lambda \rightarrow \mathfrak{p}_\lambda, \chi_{2\lambda} : \mathfrak{k}_{2\lambda} \rightarrow \mathfrak{p}_{2\lambda}$ be isomorphisms of linear spaces such that $\text{Ad}_{K_0}|_{\mathfrak{p}_\lambda} \circ \chi_\lambda = \chi_\lambda \circ \text{Ad}_{K_0}|_{\mathfrak{k}_\lambda}$ and $\text{Ad}_{K_0}|_{\mathfrak{p}_{2\lambda}} \circ \chi_{2\lambda} = \chi_{2\lambda} \circ \text{Ad}_{K_0}|_{\mathfrak{k}_{2\lambda}}$.

After the substitution $\mathfrak{p} \rightarrow \tilde{\mathfrak{p}}, \mathfrak{k} \rightarrow \mathfrak{k}_0$ we can apply the construction from section 2 for calculating generators and relations for the algebra $\text{Diff}(M_{\mathbb{S}})$. Let g_i be independent invariants of the Ad_{K_0} -action on $\tilde{\mathfrak{p}}$. Then elements $\eta \circ \mu(g_i)$ generate the algebra $\text{Diff}(M_{\mathbb{S}})$.

There are some obvious invariants of the Ad_{K_0} -action on $\tilde{\mathfrak{p}}$. First of all it is $\Lambda \in \mathfrak{a}$, since $[\mathfrak{k}_0, \mathfrak{a}] = 0$. Secondly, the Ad_{K_0} -action is isometric w.r.t. the Killing form which is proportional to the scalar product $\langle \cdot, \cdot \rangle$, so the Ad_{K_0} -action conserves the restrictions of $\langle \cdot, \cdot \rangle$ onto spaces $\mathfrak{p}_\lambda, \mathfrak{p}_{2\lambda}, \mathfrak{k}_\lambda, \mathfrak{k}_{2\lambda}$. Thirdly, the Ad_{K_0} -action conserves functions $\langle \chi_\lambda X_1, Y_1 \rangle, X_1 \in \mathfrak{k}_\lambda, Y_1 \in \mathfrak{p}_\lambda$ and $\langle \chi_{2\lambda} X_2, Y_2 \rangle, X_2 \in \mathfrak{k}_{2\lambda}, Y_2 \in \mathfrak{p}_{2\lambda}$.

Let us define χ_λ and $\chi_{2\lambda}$ by

$$\chi_\lambda X = 2[\Lambda, X] \quad X \in \mathfrak{k}_\lambda \quad \chi_{2\lambda} Y = [\Lambda, Y] \quad Y \in \mathfrak{k}_{2\lambda}. \tag{6}$$

It is obvious that $\chi_\lambda : \mathfrak{k}_\lambda \mapsto \mathfrak{p}_\lambda$ and $\chi_{2\lambda} : \mathfrak{k}_{2\lambda} \mapsto \mathfrak{p}_{2\lambda}$ are bijections. For any $k \in K_0$ it holds that $\text{Ad}_k \Lambda = \Lambda$ since $[\mathfrak{k}_0, \mathfrak{a}] = 0$, therefore from (6) we obtain: $\text{Ad}_k \circ \chi_\lambda X = 2[\Lambda, \text{Ad}_k X] = \chi_\lambda \circ \text{Ad}_k X, X \in \mathfrak{k}_\lambda$ and $\text{Ad}_k \circ \chi_{2\lambda} Y = [\Lambda, \text{Ad}_k Y] = \chi_{2\lambda} \circ \text{Ad}_k Y, Y \in \mathfrak{k}_{2\lambda}$. It is clear that $\chi_\lambda f_{\lambda,i} = e_{\lambda,i}, i = 1, \dots, q_1$ and $\chi_{2\lambda} f_{2\lambda,j} = e_{2\lambda,j}, j = 1, \dots, q_2$.

The bases

$$\left\{ \frac{1}{R} e_{\lambda,i} \right\}_{i=1}^{q_1} \quad \left\{ \frac{1}{R} f_{\lambda,i} \right\}_{i=1}^{q_1} \quad \left\{ \frac{1}{R} e_{2\lambda,j} \right\}_{j=1}^{q_2} \quad \left\{ \frac{1}{R} f_{2\lambda,j} \right\}_{j=1}^{q_2}$$

respectively in spaces $\mathfrak{p}_\lambda, \mathfrak{k}_\lambda, \mathfrak{p}_{2\lambda}, \mathfrak{k}_{2\lambda}$ are orthonormal, so one has the following generators of $\text{Diff}(M_{\mathbb{S}})$:

$$\begin{aligned} D_0 &= \eta(\Lambda) & D_1 &= \eta \left(\sum_{i=1}^{q_1} e_{\lambda,i}^2 \right) & D_2 &= \eta \left(\sum_{i=1}^{q_1} f_{\lambda,i}^2 \right) & D_3 &= \eta \left(\frac{1}{2} \sum_{i=1}^{q_1} \{e_{\lambda,i}, f_{\lambda,i}\} \right) \\ D_4 &= \eta \left(\sum_{j=1}^{q_2} e_{2\lambda,j}^2 \right) & D_5 &= \eta \left(\sum_{j=1}^{q_2} f_{2\lambda,j}^2 \right) & D_6 &= \eta \left(\frac{1}{2} \sum_{j=1}^{q_2} \{e_{2\lambda,j}, f_{2\lambda,j}\} \right) \end{aligned} \tag{7}$$

¹ By (α, \dots, ω) we denote the set of objects α, \dots, ω .

where $\{\cdot, \cdot\}$ means anticommutator. For brevity we shall omit the symbol η below. From (5) one easily obtains

$$\begin{aligned} [D_0, D_1] &= -D_3 & [D_0, D_2] &= D_3 & [D_0, D_3] &= \frac{1}{2}(D_1 - D_2) \\ [D_0, D_4] &= -2D_6 & [D_0, D_5] &= 2D_6 & [D_0, D_6] &= D_4 - D_5. \end{aligned}$$

In order to find the full system of invariants and relations in $\text{Diff}(M_{\mathbb{S}})$ we need more detailed information about the Ad_{K_0} -action on $\tilde{\mathfrak{p}}$ and commutators in \mathfrak{g} . This information will be extracted in the following sections from the models of two-point homogeneous compact Riemannian spaces.

It is easily seen that every automorphism of Lie algebra \mathfrak{g} , conserving its subalgebra \mathfrak{k}_0 , generates an automorphism of $\text{Diff}(M_{\mathbb{S}})$. From relations (4) one obtains that the map $\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$, $\sigma|_{\mathfrak{k}} = \text{id}$, $\sigma|_{\mathfrak{p}} = -\text{id}$ is the automorphism of \mathfrak{g} . It generates the automorphism of $\text{Diff}(M_{\mathbb{S}})$: $D_0 \rightarrow -D_0$, $D_1 \rightarrow D_1$, $D_2 \rightarrow D_2$, $D_3 \rightarrow -D_3$, $D_4 \rightarrow D_4$, $D_5 \rightarrow D_5$, $D_6 \rightarrow -D_6$. We shall denote it by the same symbol σ .

Another obvious automorphism is the one parametric group ζ_α of internal automorphisms, generated by the ad_Λ -action. From (5) one obtains

$$\begin{aligned} \zeta_\alpha(\Lambda) &= \Lambda & \zeta_\alpha(e_{\lambda,i}) &= \cos(\alpha/2)e_{\lambda,i} - \sin(\alpha/2)f_{\lambda,i} \\ \zeta_\alpha(f_{\lambda,i}) &= \sin(\alpha/2)e_{\lambda,i} + \cos(\alpha/2)f_{\lambda,i} & i &= 1, \dots, q_1 \\ \zeta_\alpha(e_{2\lambda,j}) &= \cos(\alpha)e_{2\lambda,j} - \sin(\alpha)f_{2\lambda,j} \\ \zeta_\alpha(f_{2\lambda,j}) &= \sin(\alpha)e_{2\lambda,j} + \cos(\alpha)f_{2\lambda,j} & j &= 1, \dots, q_2. \end{aligned}$$

Therefore,

$$\begin{aligned} \zeta_\alpha(D_0) &= D_0 & \zeta_\alpha(D_1) &= \cos^2(\alpha/2)D_1 + \sin^2(\alpha/2)D_2 - \sin(\alpha)D_3 \\ \zeta_\alpha(D_2) &= \sin^2(\alpha/2)D_1 + \cos^2(\alpha/2)D_2 + \sin(\alpha)D_3 \\ \zeta_\alpha(D_3) &= \frac{1}{2}\sin(\alpha)(D_1 - D_2) + \cos(\alpha)D_3 \\ \zeta_\alpha(D_4) &= \cos^2(\alpha)D_4 + \sin^2(\alpha)D_5 - \sin(2\alpha)D_6 \\ \zeta_\alpha(D_5) &= \sin^2(\alpha)D_4 + \cos^2(\alpha)D_5 + \sin(2\alpha)D_6 \\ \zeta_\alpha(D_6) &= \frac{1}{2}\sin(2\alpha)(D_4 - D_5) + \cos(2\alpha)D_6. \end{aligned}$$

In particular $\zeta_\pi(D_1) = D_2$, $\zeta_\pi(D_2) = D_1$, $\zeta_\pi(D_3) = -D_3$, $\zeta_\pi(D_i) = D_i$, $i = 0, 4, 5, 6$.

Proposition 1 implies that the base

$$\frac{1}{R}\Lambda \quad \frac{1}{R}e_{\lambda,i} \quad \frac{1}{R}f_{\lambda,i} \quad \frac{1}{R}e_{2\lambda,j} \quad \frac{1}{R}f_{2\lambda,j} \quad i = 1, \dots, q_1 \quad j = 1, \dots, q_2$$

in the space $\tilde{\mathfrak{p}}$ is orthonormal, therefore the operator $D^* = D_0^2 + D_1 + D_2 + D_4 + D_5$ corresponds to the Casimir operator in $U(\mathfrak{g})$ [17] (lecture 18). It means that D^* lies in the centre of the algebra $\text{Diff}(M_{\mathbb{S}})$.

Let $\pi_1 : M_{\mathbb{S}} \rightarrow M$ be the canonical projection and $\tilde{\pi}_1$ is the map of a function f on M to the function $f \circ \pi_1$ on $M_{\mathbb{S}}$. Due to the identification $\mathfrak{p} \simeq T_{x_0}M$ it is clear that the operator $(D_0^2 + D_1 + D_2) \circ \tilde{\pi}_1$ is the Laplace–Beltrami operator on M .

5. The model for the space $\mathbf{P}^n(\mathbb{H})$

Let \mathbb{H} be the quaternion algebra over the field \mathbb{R} with the base $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$, where $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$, $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$. The conjugation acts as follows: $x + y\mathbf{i} + z\mathbf{j} + t\mathbf{k} = x - y\mathbf{i} - z\mathbf{j} - t\mathbf{k}$, $x, y, z, t \in \mathbb{R}$.

Let \mathbb{H}^{n+1} be the right quaternion space and (z_1, \dots, z_{n+1}) be coordinates on it. Let $\mathbf{P}^n(\mathbb{H})$ be a factor space of the space $\mathbb{H}^{n+1} \setminus \{0\}$ with respect to the right action of the multiplicative

group $\mathbb{H}^* = \mathbb{H} \setminus \{0\}$. The set $(z_1 : \dots : z_{n+1})$ up to multiplication from the right by an arbitrary element from the group \mathbb{H}^* is the set of homogeneous coordinates for the element² $\pi(\mathbf{z})$ on the space $\mathbf{P}^n(\mathbb{H})$, where $\pi : \mathbb{H}^{n+1} \setminus \{0\} \rightarrow \mathbf{P}^n(\mathbb{H})$ is the canonical projection. Let $\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{i=1}^{n+1} \bar{x}_i y_i$, $\mathbf{x} = (x_1, \dots, x_{n+1})$, $\mathbf{y} = (y_1, \dots, y_{n+1}) \in \mathbb{H}^{n+1}$ be the standard scalar product in the space \mathbb{H}^{n+1} . Let $\mathbf{z} \in \mathbb{H}^{n+1} \setminus \{0\}$, $\xi_i \in T_{\mathbf{z}}\mathbb{H}^{n+1}$, $\zeta_i = \pi_* \xi_i \in T_{\pi(\mathbf{z})}(\mathbf{P}^n(\mathbb{H}))$, $i = 1, 2$. A metric on the space $\mathbf{P}^n(\mathbb{H})$

$$\tilde{g}|_{\mathbf{z}}(\zeta_1, \zeta_2) = (\langle \xi_1, \xi_2 \rangle \langle \mathbf{z}, \mathbf{z} \rangle - \langle \xi_1, \mathbf{z} \rangle \langle \mathbf{z}, \xi_2 \rangle) / \langle \mathbf{z}, \mathbf{z} \rangle^2 \tag{8}$$

is the analogue for the metric with a constant sectional curvature on the space $\mathbf{P}^n(\mathbb{R})$ and the metric with a constant holomorphic sectional curvature on the space $\mathbf{P}^n(\mathbb{C})$. The real part of the metric (8) is a Riemannian metric on the space $\mathbf{P}^n(\mathbb{H})$:

$$g = 4R^2 \operatorname{Re} \tilde{g}. \tag{9}$$

The normalizing factor in (9) is chosen due to the following reasons. The space $\mathbf{P}^1(\mathbb{H})$ with this metric is the sphere \mathbf{S}^4 with the standard metric of the constant sectional curvature R^{-2} . To see this we can consider a homeomorphism $\nu : \mathbf{P}^1(\mathbb{H}) \rightarrow \overline{\mathbb{H}} \cong \mathbf{S}^4$, $\nu(z_1, z_2) = z_1(z_2)^{-1} = z \in \overline{\mathbb{H}}$, where $\overline{\mathbb{H}}$ is the quaternion space completed with the point at infinity. For $n = 1$ formula (9) has the form

$$g = 4R^2 \frac{(d\bar{z}_1 dz_1 + d\bar{z}_2 dz_2)(|z_1|^2 + |z_2|^2) - (d\bar{z}_1 \cdot z_1 + d\bar{z}_2 \cdot z_2)(\bar{z}_1 dz_1 + \bar{z}_2 dz_2)}{(|z_1|^2 + |z_2|^2)^2}. \tag{10}$$

Using the formula $|z_2|^2 dz_1 - z_1 \bar{z}_2 dz_2 = |z_2|^2 (dz)z_2$ by direct calculations we can reduce expression (10) to the form

$$g = \frac{4R^2 dz d\bar{z}}{(1 + |z|^2)^2}$$

which is the metric with the constant sectional curvature R^{-2} on the sphere \mathbf{S}^4 .

The left action of the group $U_{\mathbb{H}}(n + 1)$, consisting of quaternion matrices A of the size $(n + 1) \times (n + 1)$ such that $\bar{A}^T A = E$, conserves the scalar product $\langle \cdot, \cdot \rangle$ in the space \mathbb{H}^{n+1} , $\dim_{\mathbb{R}} U_{\mathbb{H}}(n + 1) = (2n + 3)(n + 1)$. If we write every quaternion coordinates in \mathbb{H}^{n+1} as a pair of complex numbers, then the group $U_{\mathbb{H}}(n + 1)$ becomes the symplectic group $Sp(n + 1)$.

Left and right multiplications always commute, so the left action of the group $U_{\mathbb{H}}(n + 1)$ is correctly defined also on the space $\mathbf{P}^n(\mathbb{H})$. Obviously, it is transitive and conserves the metric g . The stationary subgroup of the point from the space $\mathbf{P}^n(\mathbb{H})$ with the homogeneous coordinates $(1, 0, \dots, 0)$ is the group $U_{\mathbb{H}}(n)U_{\mathbb{H}}(1)$, where the group $U_{\mathbb{H}}(n)$ acts on the last n coordinates, and the group $U_{\mathbb{H}}(1)$ acts by the left multiplication of all homogeneous coordinates by quaternions with the unit norm. All stationary subgroups on a homogeneous space are conjugated and hence isomorphic. Therefore $\mathbf{P}^n(\mathbb{H}) = U_{\mathbb{H}}(n + 1) / (U_{\mathbb{H}}(n)U_{\mathbb{H}}(1))$.

The Lie algebra $\mathfrak{u}_{\mathbb{H}}(n + 1)$ consists of quaternion matrices A of the size $(n + 1) \times (n + 1)$ such that $\bar{A}^T = -A$. Let E_{kj} be the matrix of the size $(n + 1) \times (n + 1)$ with the unique nonzero element equalling 1, located at the intersection of the k th row and the j th column. Choose the base for the algebra $\mathfrak{u}_{\mathbb{H}}(n + 1)$ as

$$\begin{aligned} \Psi_{kj} &= \frac{1}{2}(E_{kj} - E_{jk}) & 1 \leq k < j \leq n + 1 & & \Upsilon_{kj} &= \frac{\mathbf{i}}{2}(E_{kj} + E_{jk}) \\ \Omega_{kj} &= \frac{\mathbf{j}}{2}(E_{kj} + E_{jk}) & \Theta_{kj} &= \frac{\mathbf{k}}{2}(E_{kj} + E_{jk}) & 1 \leq k \leq j \leq n + 1. \end{aligned} \tag{11}$$

² To distinguish the point $x \in M$ from their coordinates we shall single out it by the bold type.

The commutative relations for these elements are

$$\begin{aligned}
 [\Psi_{kj}, \Psi_{ml}] &= \frac{1}{2}(\delta_{jm}\Psi_{kl} - \delta_{km}\Psi_{jl} + \delta_{kl}\Psi_{jm} - \delta_{jl}\Psi_{km}) \\
 [\Psi_{kj}, \Upsilon_{ml}] &= \frac{1}{2}(\delta_{jm}\Upsilon_{kl} - \delta_{km}\Upsilon_{jl} + \delta_{lj}\Upsilon_{km} - \delta_{lk}\Upsilon_{jm}) \\
 [\Upsilon_{kj}, \Upsilon_{ml}] &= \frac{1}{2}(\delta_{jm}\Psi_{lk} + \delta_{km}\Psi_{lj} + \delta_{kl}\Psi_{mj} + \delta_{jl}\Psi_{mk}) \\
 [\Upsilon_{kj}, \Omega_{ml}] &= \frac{1}{2}(\delta_{jm}\Theta_{lk} + \delta_{km}\Theta_{lj} + \delta_{kl}\Theta_{mj} + \delta_{jl}\Theta_{mk})
 \end{aligned} \tag{12}$$

plus the analogous equalities, obtained from the latter three relations by the cyclic permutation $\Upsilon \rightarrow \Omega \rightarrow \Theta \rightarrow \Upsilon$, where $\Psi_{kj} = -\Psi_{jk}$, $\Psi_{kk} = 0$, $\Upsilon_{kj} = \Upsilon_{jk}$, $\Omega_{kj} = \Omega_{jk}$, $\Theta_{kj} = \Theta_{jk}$.

6. Algebras $\text{Diff}(\mathbf{P}^n(\mathbb{H})_{\mathbb{S}})$ and $\text{Diff}(\mathbf{H}^n(\mathbb{H})_{\mathbb{S}})$

Consider now the total space of unit spheres bundle $\mathbf{P}^n(\mathbb{H})_{\mathbb{S}}$ over the space $\mathbf{P}^n(\mathbb{H})$. Let (\mathbf{z}, ζ) , where $\mathbf{z} \in \mathbf{P}^n(\mathbb{H})$, $\zeta \in T_{\mathbf{z}}\mathbf{P}^n(\mathbb{H})$ be a general point of the space $\mathbf{P}^n(\mathbb{H})_{\mathbb{S}}$. Due to the isomorphism $\mathbf{P}^1(\mathbb{H}) \cong \mathbf{S}^4$ we assume here $n \geq 2$.

Let $\tilde{\mathbf{z}}_0 = (1, 0, \dots, 0) \in \mathbb{H}^{n+1}$, an element $\xi_0 \in T_{\tilde{\mathbf{z}}_0}\mathbb{H}^{n+1} \cong \mathbb{H}^{n+1}$ has coordinates $(0, 1, 0, \dots, 0)$. Put $\mathbf{z}_0 = \pi\tilde{\mathbf{z}}_0$, $\zeta_0 = \pi_*\xi_0 \in T_{\mathbf{z}_0}\mathbf{P}^n(\mathbb{H})$.

The stationary subgroup K_0 of the group $U_{\mathbb{H}}(n+1)$, corresponding to the point (\mathbf{z}_0, ζ_0) is generated by the group $K_1 = U_{\mathbb{H}}(n-1)$, acting on the last $(n-1)$ th homogeneous coordinates and by the group $K_2 = U_{\mathbb{H}}(1)$, acting by the left multiplication of all homogeneous coordinates by quaternions with unit norm. The algebra \mathfrak{k}_0 of the group K_0 (corresponding to proposition 1) is $(2n^2 - 3n + 4)$ dimensional and is generated by elements (11) with $3 \leq k \leq j \leq n+1$ and the elements:

$$\sum_{k=1}^{n+1} \Upsilon_{kk} \quad \sum_{k=1}^{n+1} \Omega_{kk} \quad \sum_{k=1}^{n+1} \Theta_{kk}.$$

Choose the complementary subspace $\tilde{\mathfrak{p}}$ to the subalgebra \mathfrak{k}_0 in the algebra $\mathfrak{g} = u_{\mathbb{H}}(n+1)$ as the linear hull of elements:

$$\begin{aligned}
 &\Psi_{1k} \quad \Upsilon_{1k} \quad \Omega_{1k} \quad \Theta_{1k} \quad 2 \leq k \leq n+1 \quad \Psi_{2k} \quad \Upsilon_{2k} \quad \Omega_{2k} \quad \Theta_{2k} \quad 3 \leq k \leq n+1 \\
 \Upsilon_* &= \frac{\mathbf{i}}{2}(E_{11} - E_{22}) \quad \Omega_* = \frac{\mathbf{j}}{2}(E_{11} - E_{22}) \quad \Theta_* = \frac{\mathbf{k}}{2}(E_{11} - E_{22}).
 \end{aligned} \tag{13}$$

Taking into account relations (12) it is easily obtained that the expansion $u_{\mathbb{H}}(n+1) = \tilde{\mathfrak{p}} \oplus \mathfrak{k}_0$ is reductive, i.e. $[\tilde{\mathfrak{p}}, \mathfrak{k}_0] \subset \tilde{\mathfrak{p}}$.

It is readily seen from (12) that setting

$$\begin{aligned}
 \Lambda &= -\Psi_{12} & e_{\lambda, k-2} &= \Psi_{1k} & e_{\lambda, n-3+k} &= \Upsilon_{1k} & e_{\lambda, 2n-4+k} &= \Omega_{1k} \\
 e_{\lambda, 3n-5+k} &= \Theta_{1k} & f_{\lambda, k-2} &= -\Psi_{2k} & f_{\lambda, n-3+k} &= -\Upsilon_{2k} & f_{\lambda, 2n-4+k} &= -\Omega_{2k} \\
 f_{\lambda, 3n-5+k} &= -\Theta_{2k} & k &= 3, \dots, n+1 & e_{2\lambda, 1} &= \Upsilon_{12} & e_{2\lambda, 2} &= \Omega_{12} \\
 e_{2\lambda, 3} &= \Theta_{12} & f_{2\lambda, 1} &= \Upsilon_* & f_{2\lambda, 2} &= \Omega_* & f_{2\lambda, 3} &= \Theta_*
 \end{aligned} \tag{14}$$

we obtain the base from proposition 1 for $q_1 = 4n - 4$, $q_2 = 3$.

Now we find the full set of independent Ad_{K_0} -invariants in $S(\tilde{\mathfrak{p}})$. According to section 4, the expansion $\tilde{\mathfrak{p}} = \mathfrak{a} \oplus \mathfrak{k}_{\lambda} \oplus \mathfrak{k}_{2\lambda} \oplus \mathfrak{p}_{\lambda} \oplus \mathfrak{p}_{2\lambda}$ is invariant w.r.t. the Ad_{K_0} -action. In the space \mathfrak{a} the K_0 -action is trivial that gives the invariant $D_0 = \Lambda \in \mu(\mathfrak{p}^{K_0})$, already found in section 4.

From formulae (14) we see that the space $\mathfrak{p}_\lambda \cong \mathbb{H}^{n-1}$ consists of matrices of the form

$$\begin{pmatrix} 0 & -a^* \\ a & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & -\bar{a}_1 & \dots & -\bar{a}_{n-1} \\ 0 & 0 & 0 & \dots & 0 \\ a_1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & 0 & 0 & \dots & 0 \end{pmatrix} \quad a_1, \dots, a_{n-1} \in \mathbb{H}.$$

Likewise, the space $\mathfrak{k}_\lambda \cong \mathbb{H}^{n-1}$ consists of matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -b^* \\ 0 & b & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -\bar{b}_1 & \dots & -\bar{b}_{n-1} \\ 0 & b_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n-1} & 0 & \dots & 0 \end{pmatrix} \quad b_1, \dots, b_{n-1} \in \mathbb{H}.$$

Due to the formula

$$\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} 0 & -a^* \\ a & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & U^* \end{pmatrix} = \begin{pmatrix} 0 & -(Ua)^* \\ Ua & 0 \end{pmatrix} \quad U \in U_{\mathbb{H}}(n-1) \quad a \in \mathbb{H}^{n-1}$$

the action of the group K_1 on the space \mathfrak{p}_λ is equivalent to the standard action of the group $U_{\mathbb{H}}(n-1)$ on the space $\mathbb{H}^{n-1} : a \rightarrow Ua$. In the space \mathfrak{k}_λ the action of K_1 is similar: $b \rightarrow Ub$.

The standard action of the group $U_{\mathbb{H}}(n-1)$ on the space \mathbb{H}^{n-1} has one independent real invariant: $\langle \mathbf{z}, \mathbf{z} \rangle, \mathbf{z} \in \mathbb{H}^{n-1}$, and the diagonal action of $U_{\mathbb{H}}(n-1)$ on the space $\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \cong \mathbb{H}^{n-1} \oplus \mathbb{H}^{n-1}$ has six (independent iff $n \geq 3$) real invariants:

$$\langle \mathbf{z}_1, \mathbf{z}_1 \rangle \in \mathbb{R} \quad \langle \mathbf{z}_2, \mathbf{z}_2 \rangle \in \mathbb{R} \quad \langle \mathbf{z}_1, \mathbf{z}_2 \rangle \in \mathbb{H} \cong \mathbb{R}^4 \quad \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{H}^{n-1}. \tag{15}$$

Denote the corresponding elements from $\mu(\tilde{\mathfrak{p}}^{K_1}) \in U(\mathfrak{g})^{K_1}$ in the following way:

$$\begin{aligned} D_1 &= \sum_{k=3}^{n+1} (\Psi_{1k}^2 + \Upsilon_{1k}^2 + \Omega_{1k}^2 + \Theta_{1k}^2) & D_2 &= \sum_{k=3}^{n+1} (\Psi_{2k}^2 + \Upsilon_{2k}^2 + \Omega_{2k}^2 + \Theta_{2k}^2) \\ D_3 &= -\frac{1}{2} \sum_{k=3}^{n+1} (\{\Psi_{1k}, \Psi_{2k}\} + \{\Upsilon_{1k}, \Upsilon_{2k}\} + \{\Omega_{1k}, \Omega_{2k}\} + \{\Theta_{1k}, \Theta_{2k}\}) \\ \square_1 &= \frac{1}{2} \sum_{k=3}^{n+1} (-\{\Psi_{1k}, \Upsilon_{2k}\} + \{\Psi_{2k}, \Upsilon_{1k}\} + \{\Theta_{1k}, \Omega_{2k}\} - \{\Theta_{2k}, \Omega_{1k}\}) \\ \square_2 &= \frac{1}{2} \sum_{k=3}^{n+1} (-\{\Psi_{1k}, \Omega_{2k}\} + \{\Psi_{2k}, \Omega_{1k}\} + \{\Upsilon_{1k}, \Theta_{2k}\} - \{\Upsilon_{2k}, \Theta_{1k}\}) \\ \square_3 &= \frac{1}{2} \sum_{k=3}^{n+1} (-\{\Psi_{1k}, \Theta_{2k}\} + \{\Psi_{2k}, \Theta_{1k}\} + \{\Omega_{1k}, \Upsilon_{2k}\} - \{\Omega_{2k}, \Upsilon_{1k}\}). \end{aligned} \tag{16}$$

If $n = 2$, then there is the unique independent relation between invariants (15):

$$|\langle \mathbf{z}_1, \mathbf{z}_2 \rangle|^2 = |\bar{z}_1 z_2|^2 = |z_1|^2 |z_2|^2 = \langle \mathbf{z}_1, \mathbf{z}_1 \rangle \langle \mathbf{z}_2, \mathbf{z}_2 \rangle, \mathbf{z}_1 = z_1, \mathbf{z}_2 = z_2 \in \mathbb{H}. \tag{17}$$

If we write this identity in coordinates, then we will obtain the well-known Euler identity which is the key ingredient in the proof of the Lagrange theorem from number theory: *if two integers have the form $a^2 + b^2 + c^2 + d^2$, $a, b, c, d \in \mathbb{Z}$, then their product has the same form.*

The elements D_1, D_2, D_3 , already found in section 4, are invariant w.r.t. the action of the whole group K_0 , therefore they correspond to operators of the second order from

$\text{Diff}(\mathbf{P}^n(\mathbb{H})_{\mathbb{S}})$. The elements $\square_1, \square_2, \square_3$ are not invariant w.r.t. the action of the group $K_2 \cong U_{\mathbb{H}}(1)$. Obviously, the K_2 -action on the linear hull of elements $\square_1, \square_2, \square_3$ is equivalent to the well-known action of the group $SO(3) \cong U_{\mathbb{H}}(1)/(1, -1)$ on the space \mathbb{H}' of pure imaginary quaternions:

$$x \rightarrow qx\bar{q} \quad x \in \mathbb{H}' \quad q \in U_{\mathbb{H}}(1)$$

after the identification $\square_1 \leftrightarrow \mathbf{i}, \square_2 \leftrightarrow \mathbf{j}, \square_3 \leftrightarrow \mathbf{k}$.

The K_2 -action on three-dimensional spaces $\mathfrak{p}_{2\lambda}, \mathfrak{k}_{2\lambda}$ is the same after the identification $\Upsilon_{12}, \Upsilon_* \leftrightarrow \mathbf{i}; \Omega_{12}, \Omega_* \leftrightarrow \mathbf{j}; \Theta_{12}, \Theta_* \leftrightarrow \mathbf{k}$; while the K_1 -action on these spaces is trivial. Thus we find invariants of diagonal action of the group $SO(3)$ on the space $\mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$. It is clear that there are $6 = 9 - 3$ such independent invariants:

$$\langle \mathbf{x}, \mathbf{x} \rangle, \quad \langle \mathbf{y}, \mathbf{y} \rangle, \quad \langle \mathbf{z}, \mathbf{z} \rangle, \quad \langle \mathbf{x}, \mathbf{y} \rangle, \quad \langle \mathbf{x}, \mathbf{z} \rangle, \quad \langle \mathbf{z}, \mathbf{y} \rangle, \quad \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$$

and invariant $\langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle \equiv \langle \mathbf{x}, \mathbf{y} \times \mathbf{z} \rangle$ algebraically connected with the first six:

$$\langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle^2 = \mathbf{x}^2 \mathbf{y}^2 \mathbf{z}^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle \langle \mathbf{x}, \mathbf{z} \rangle \langle \mathbf{y}, \mathbf{z} \rangle - \mathbf{x}^2 \langle \mathbf{y}, \mathbf{z} \rangle^2 - \mathbf{y}^2 \langle \mathbf{x}, \mathbf{z} \rangle^2 - \mathbf{z}^2 \langle \mathbf{x}, \mathbf{y} \rangle^2 \tag{18}$$

where $\mathbf{y} \times \mathbf{z}$ is the standard vector product in \mathbb{R}^3 . Relation (18) can be verified using the well-known formulae: $\langle \mathbf{x}, \mathbf{y} \rangle^2 = \mathbf{x}^2 \mathbf{y}^2 - \langle \mathbf{x} \times \mathbf{y} \rangle^2$ and $\mathbf{x} \times (\mathbf{y} \times \mathbf{x}) = \langle \mathbf{x}, \mathbf{z} \rangle \mathbf{y} - \langle \mathbf{x}, \mathbf{y} \rangle \mathbf{z}$.

It gives the following invariants from $U(\mathfrak{g})^{K_0}$:

$$\begin{aligned} D_4 &= \Upsilon_{12}^2 + \Omega_{12}^2 + \Theta_{12}^2 & D_5 &= \Upsilon_*^2 + \Omega_*^2 + \Theta_*^2 \\ D_6 &= \frac{1}{2}(\{\Upsilon_{12}, \Upsilon_*\} + \{\Omega_{12}, \Omega_*\} + \{\Theta_{12}, \Theta_*\}) \\ D_7 &= \frac{1}{2}(\{\square_1, \Upsilon_{12}\} + \{\square_2, \Omega_{12}\} + \{\square_3, \Theta_{12}\}) \\ D_8 &= \frac{1}{2}(\{\square_1, \Upsilon_*\} + \{\square_2, \Omega_*\} + \{\square_3, \Theta_*\}) & D_9 &= \square_1^2 + \square_2^2 + \square_3^2 \\ D_{10} &= \square_1 \Omega_{12} \Theta_* - \square_1 \Omega_* \Theta_{12} + \square_2 \Upsilon_* \Theta_{12} - \square_2 \Upsilon_{12} \Theta_* + \square_3 \Omega_* \Upsilon_{12} - \square_3 \Omega_{12} \Upsilon_* \end{aligned} \tag{19}$$

Here we took into account that every three factors from all summands in the last expression pairwise commute. The invariants D_4, D_5, D_6 correspond to the general case, considered in section 4.

In fact invariants D_7, D_8, D_9 and D_{10} are not in $\mu(\mathfrak{p}^{K_0})$ because they are not symmetric w.r.t. all transpositions of their factors of the first degree. After complete symmetrization we can obtain invariants from $\mu(\mathfrak{p}^{K_0}) : \tilde{D}_k \equiv D_k + D_k^* \pmod{U(\mathfrak{g})\mathfrak{k}^{K_0}}, k = 7, 8, 9, 10$, where D_k^* are elements from $U(\mathfrak{g})^{K_0}$ with $\text{deg } D_k^* < \text{deg } D_k$. For convenience we will use elements D_k instead of $\tilde{D}_k, k = 7, 8, 9, 10$.

Thus operators D_0, \dots, D_{10} generate the algebra $\text{Diff}(\mathbf{P}^n(\mathbb{H})_{\mathbb{S}})$.

The degrees of the generators are as follows:

$$\begin{aligned} \text{deg}(D_0) &= 1 & \text{deg}(D_1) &= \text{deg}(D_2) = \text{deg}(D_3) = \text{deg}(D_4) = \text{deg}(D_5) = \text{deg}(D_6) = 2 \\ \text{deg}(D_7) &= \text{deg}(D_8) = 3 & \text{deg}(D_9) &= \text{deg}(D_{10}) = 4. \end{aligned} \tag{20}$$

In the model of the space $\mathbf{P}^n(\mathbb{H})$ we can transpose the coordinates z_1 and z_2 . The operators $D_3, D_4, D_5, D_8, D_9, D_{10}$ are symmetric (invariant) w.r.t. this transposition and the operators $D_0, \square_1, \square_2, \square_3, D_6, D_7$ are skew symmetric. The operators D_1 and D_2 turn into each other under this transposition.

It is easily verified that automorphisms ζ_α, σ acts on $\square_i, D_7, \dots, D_{10}, i = 1, 2, 3$ as

$$\begin{aligned} \zeta_\alpha(\square_i) &= \square_i & i &= 1, 2, 3 & \zeta_\alpha(D_7) &= \cos(\alpha)D_7 - \sin(\alpha)D_8 \\ \zeta_\alpha(D_8) &= \sin(\alpha)D_7 + \cos(\alpha)D_8 & \zeta_\alpha(D_9) &= D_9 & \zeta_\alpha(D_{10}) &= D_{10} & \sigma(\square_i) &= -\square_i \\ i &= 1, 2, 3 & \sigma(D_7) &= D_7 & \sigma(D_8) &= -D_8 & \sigma(D_9) &= D_9 & \sigma(D_{10}) &= D_{10}. \end{aligned}$$

Taking into account their action on other generators (see section 4) we obtain that the transposition of z_1 and z_2 is equivalent to the composition $\sigma \circ \zeta_\pi$.

In order to get the generators of the algebra $\text{Diff}(\mathbf{H}^n(\mathbb{H})_{\mathbb{S}})$ one can use remark 1, formula (14) and make the formal substitution:

$$\begin{array}{llll} \Lambda \rightarrow \mathbf{i}\Lambda & \Psi_{1k} \rightarrow \mathbf{i}\Psi_{1k} & \Upsilon_{1k} \rightarrow \mathbf{i}\Upsilon_{1k} & \Omega_{1k} \rightarrow \mathbf{i}\Omega_{1k} \\ \Theta_{1k} \rightarrow \mathbf{i}\Theta_{1k} & \Upsilon_{12} \rightarrow \mathbf{i}\Upsilon_{12} & \Omega_{12} \rightarrow \mathbf{i}\Omega_{12} & \Theta_{12} \rightarrow \mathbf{i}\Theta_{12} \\ \Psi_{2k} \rightarrow \Psi_{2k} & \Upsilon_{2k} \rightarrow \Upsilon_{2k} & \Omega_{2k} \rightarrow \Omega_{2k} & \Theta_{2k} \rightarrow \Theta_{2k} \\ \Upsilon_* \rightarrow \Upsilon_* & \Omega_* \rightarrow \Omega_* & \Theta_* \rightarrow \Theta_* & k = 3, \dots, n+1. \end{array}$$

This substitution produces the following substitution for the generators D_0, \dots, D_{10} :

$$\begin{array}{llll} D_0 \rightarrow \mathbf{i}\bar{D}_0 & D_1 \rightarrow -\bar{D}_1 & D_2 \rightarrow \bar{D}_2 & D_3 \rightarrow \mathbf{i}\bar{D}_3 \\ D_4 \rightarrow -\bar{D}_4 & D_5 \rightarrow \bar{D}_5 & D_6 \rightarrow \mathbf{i}\bar{D}_6 & D_7 \rightarrow -\bar{D}_7 \\ D_8 \rightarrow \mathbf{i}\bar{D}_8 & D_9 \rightarrow -\bar{D}_9 & D_{10} \rightarrow -\bar{D}_{10}. \end{array} \quad (21)$$

The operators $\bar{D}_0, \dots, \bar{D}_{10}$ generate the algebra $\text{Diff}(\mathbf{H}^n(\mathbb{H})_{\mathbb{S}})$.

7. Relations in algebras $\text{Diff}(\mathbf{P}^n(\mathbb{H})_{\mathbb{S}})$ and $\text{Diff}(\mathbf{H}^n(\mathbb{H})_{\mathbb{S}})$

Here we find the independent relations in $\text{Diff}(\mathbf{P}^n(\mathbb{H})_{\mathbb{S}})$ for its generators D_0, \dots, D_{10} . They are of two types (see section 2). The first type is commutative relations, because a commutator of two differential operator of orders m_1 and m_2 is an operator of an order $m_3 \leq m_1 + m_2 - 1$. It gives $11(11-1)/2 = 55$ relations. If $n \geq 3$ due to (18) the second type consists of only one independent relation of the form

$$D_{10}^2 - D_4 D_5 D_9 - 2D_6 D_7 D_8 + D_9 D_6^2 + D_4 D_8^2 + D_5 D_7^2 = D' \quad (22)$$

where D' is an operator of order ≤ 7 , which is polynomial in D_0, \dots, D_{10} . If $n = 2$ formula (17) gives another independent relation of the form

$$\frac{1}{2}\{D_1, D_2\} - D_3^2 - D_9 = D''$$

where D'' is an operator of order ≤ 3 , polynomial in D_0, \dots, D_8 . The direct calculations give $D'' = D_1 + D_2$, therefore in the case $n = 2$ we have the additional relations:

$$\frac{1}{2}\{D_1, D_2\} - D_3^2 - D_9 = D_1 + D_2. \quad (23)$$

For $n = 2$ using this relation we can exclude the element D_9 from the list of generators.

In principle, all relations can be obtained by straightforward calculations in $U(\mathfrak{g})$ modulo $(U(\mathfrak{g})\mathfrak{k})^{K_0}$, but these calculations became too cumbersome to write all of them here. In appendix A there is an example of deriving some commutative relations. After getting some commutative relations by direct calculations it is possible to get some other ones (see appendix A) using the Jacobi identity:

$$[D_i, [D_j, D_k]] + [D_k, [D_i, D_j]] + [D_j, [D_k, D_i]] = 0$$

which is valid, in particular, in every associative algebra. This identity gives also a tool for checking the commutative relations already found. Below there are all 55 commutative relations in lexicographic order. Relation (22) becomes too difficult to obtain in a similar way. Maybe we need computer algebra calculations to obtain the explicit expression for D'

$$\begin{aligned}
[D_0, D_1] &= -D_3 & [D_0, D_2] &= D_3 & [D_0, D_3] &= \frac{1}{2}(D_1 - D_2) \\
[D_0, D_4] &= -2D_6 & [D_0, D_5] &= 2D_6 & [D_0, D_6] &= D_4 - D_5 \\
[D_0, D_7] &= -D_8 & [D_0, D_8] &= D_7 & [D_0, D_9] &= 0 \\
[D_0, D_{10}] &= 0 & [D_1, D_2] &= -\{D_0, D_3\} - 2D_7 \\
[D_1, D_3] &= -\frac{1}{2}\{D_0, D_1\} + D_8 + n(n-1)D_0 \\
[D_1, D_4] &= 2D_7 & [D_1, D_5] &= 0 & [D_1, D_6] &= D_8 \\
[D_1, D_7] &= -\frac{1}{2}\{D_3, D_6\} - \frac{1}{2}\{D_1, D_4\} + \frac{3}{8}(D_1 - D_2) + D_9 + D_{10} + n(n-1)D_4 \\
[D_1, D_8] &= -\frac{1}{2}\{D_3, D_5\} - \frac{1}{2}\{D_1, D_6\} + \frac{3}{4}D_3 + n(n-1)D_6 \\
[D_1, D_9] &= -\{D_3, D_8\} - \{D_1, D_7\} - \frac{3}{4}\{D_0, D_3\} + 2(n - \frac{3}{2})(n + \frac{1}{2})D_7 \\
[D_1, D_{10}] &= \frac{1}{2}\{D_6, D_8\} - \frac{1}{2}\{D_5, D_7\} + \frac{3}{8}\{D_0, D_3\} + \frac{1}{2}D_7 \\
[D_2, D_3] &= \frac{1}{2}\{D_0, D_2\} + D_8 - n(n-1)D_0 & [D_2, D_4] &= -2D_7 \\
[D_2, D_5] &= 0 & [D_2, D_6] &= -D_8 \\
[D_2, D_7] &= -\frac{1}{2}\{D_3, D_6\} + \frac{1}{2}\{D_2, D_4\} + \frac{3}{8}(D_1 - D_2) - D_9 - D_{10} - n(n-1)D_4 \\
[D_2, D_8] &= -\frac{1}{2}\{D_3, D_5\} + \frac{1}{2}\{D_2, D_6\} + \frac{3}{4}D_3 - n(n-1)D_6 \\
[D_2, D_9] &= -\{D_3, D_8\} + \{D_2, D_7\} + \frac{3}{4}\{D_0, D_3\} - 2(n - \frac{3}{2})(n + \frac{1}{2})D_7 \\
[D_2, D_{10}] &= -\frac{1}{2}\{D_6, D_8\} + \frac{1}{2}\{D_5, D_7\} - \frac{3}{8}\{D_0, D_3\} - \frac{1}{2}D_7 & [D_3, D_4] &= 0 \\
[D_3, D_5] &= 2D_8 & [D_3, D_6] &= D_7 & [D_3, D_7] &= -\frac{1}{4}\{D_1 + D_2, D_6\} + n(n-1)D_6 \\
[D_3, D_8] &= -\frac{1}{4}\{D_1 + D_2, D_5\} + n(n-1)D_5 + D_9 + D_{10} \\
[D_3, D_9] &= -\frac{1}{2}\{D_1 + D_2, D_8\} + \frac{3}{8}\{D_0, D_1 - D_2\} + 2(n - \frac{3}{2})(n + \frac{1}{2})D_8 \\
[D_3, D_{10}] &= \frac{1}{2}\{D_6, D_7\} - \frac{1}{2}\{D_4, D_8\} - \frac{3}{16}\{D_0, D_1 - D_2\} + \frac{1}{2}D_8 \\
[D_4, D_5] &= -2\{D_0, D_6\} & [D_4, D_6] &= -\{D_0, D_4\} + \frac{3}{2}D_0 \\
[D_4, D_7] &= \frac{1}{2}\{D_1 - D_2, D_4\} + \frac{3}{4}(D_2 - D_1) & [D_4, D_8] &= \frac{1}{2}\{D_1 - D_2, D_6\} - \{D_0, D_7\} \\
[D_4, D_9] &= \{D_1 - D_2, D_7\} & [D_4, D_{10}] &= 0 & [D_5, D_6] &= \{D_0, D_5\} - \frac{3}{2}D_0 \\
[D_5, D_7] &= \{D_3, D_6\} + \{D_0, D_8\} & [D_5, D_8] &= \{D_3, D_5\} - \frac{3}{2}D_3 \\
[D_5, D_9] &= 2\{D_3, D_8\} & [D_5, D_{10}] &= 0 \\
[D_6, D_7] &= \frac{1}{4}\{D_1 - D_2, D_6\} + \frac{1}{2}\{D_3, D_4\} + \frac{1}{2}\{D_0, D_7\} - \frac{3}{4}D_3 \\
[D_6, D_8] &= \frac{1}{4}\{D_1 - D_2, D_5\} + \frac{1}{2}\{D_3, D_6\} - \frac{1}{2}\{D_0, D_8\} + \frac{3}{8}(D_2 - D_1) \\
[D_6, D_9] &= \frac{1}{2}\{D_1 - D_2, D_8\} + \{D_3, D_7\} & [D_6, D_{10}] &= 0 \\
[D_7, D_8] &= \frac{1}{4}\{D_1 - D_2, D_8\} - \frac{1}{2}\{D_3, D_7\} + \frac{3}{16}\{D_0, D_1 + D_2\} \\
&& - \frac{1}{2}\{D_0, D_9 + D_{10}\} - \frac{3}{4}n(n-1)D_0 \\
[D_7, D_9] &= \frac{1}{4}\{D_3, D_6\} + \frac{1}{8}\{D_1 - D_2, D_4\} + \frac{1}{2}\{D_1 - D_2, D_9 + D_{10}\} - \frac{3}{8}(D_1^2 - D_2^2) \\
&& + \frac{3}{4}(n^2 - n - \frac{1}{4})(D_1 - D_2) \\
[D_7, D_{10}] &= \frac{1}{4}\{D_2 - D_1, D_6^2\} - \frac{1}{4}\{\{D_0, D_7\}, D_6\} + \frac{1}{4}\{\{D_0, D_4\}, D_8\} + \frac{1}{8}\{\{D_1 - D_2, D_5\}, D_4\} \\
&& - \frac{1}{4}\{D_3, D_6\} + \frac{1}{8}\{D_2 - D_1, 3D_4 + D_5\} - \frac{1}{2}\{D_0, D_8\} + \frac{15}{32}(D_1 - D_2) \\
[D_8, D_9] &= \frac{1}{8}\{D_1 - D_2, D_6\} + \frac{1}{4}\{D_3, D_5\} - \frac{3}{8}\{D_3, D_1 + D_2\} \\
&& + \{D_3, D_9 + D_{10}\} + \frac{3}{2}(n^2 - n - \frac{1}{4})D_3
\end{aligned}$$

$$\begin{aligned}
[D_8, D_{10}] &= -\frac{1}{4}\{\{D_3, D_6\}, D_6\} + \frac{1}{4}\{\{D_0, D_6\}, D_8\} - \frac{1}{4}\{\{D_0, D_5\}, D_7\} + \frac{1}{4}\{\{D_3, D_5\}, D_4\} \\
&\quad - \frac{1}{2}\{D_3, D_5\} - \frac{1}{4}\{D_3, D_4\} + \frac{1}{4}\{D_0, D_7\} + \frac{9}{16}D_3 \\
[D_9, D_{10}] &= \frac{1}{4}\{-\{D_6, D_8\} + \{D_5, D_7\}, D_1 - D_2\} + \frac{1}{2}\{\{D_3, D_8\}, D_4\} \\
&\quad - \frac{1}{2}\{\{D_3, D_6\}, D_7\} + \frac{1}{4}\{D_2 - D_1, D_7\} - \frac{1}{2}\{D_3, D_8\}.
\end{aligned} \tag{24}$$

It is interesting that the operators D_9 and D_{10} arise in the right-hand sides of these relations only in the combination $D_9 + D_{10}$.

Using relations (24) it is not difficult to verify that the operator $D^* = D_0^2 + D_1 + D_2 + D_4 + D_5$ lies in the centre of the algebra $\text{Diff}(\mathbf{P}^n(\mathbb{H})_{\mathbb{S}})$ in accordance with section 4.

Using substitution (21) one can obtain from (24) the commutative relations for the algebra $\text{Diff}(\mathbf{H}^n(\mathbb{H})_{\mathbb{S}})$.

The analogue for the operator D^* from the centre of the algebra now becomes $\bar{D}^* = \bar{D}_0^2 + \bar{D}_1 - \bar{D}_2 + \bar{D}_4 - \bar{D}_5 \in \text{Diff}(\mathbf{H}^n(\mathbb{H})_{\mathbb{S}})$. In the case $n = 2$ the additional relations (23) become

$$\frac{1}{2}\{\bar{D}_1, \bar{D}_2\} - \bar{D}_3^2 - \bar{D}_9 = \bar{D}_1 - \bar{D}_2.$$

8. Algebras $\text{Diff}(\mathbf{P}^n(\mathbb{C})_{\mathbb{S}})$ and $\text{Diff}(\mathbf{H}^n(\mathbb{C})_{\mathbb{S}})$

8.1. The model for the space $\mathbf{P}^n(\mathbb{C})$

Taking the factor space of $\mathbb{C}^{n+1} \setminus \{0\}$ w.r.t the action of the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ (due to the commutativity of the complex multiplication it makes no difference left or right), we obtain the complex projective space $\mathbf{P}^n(\mathbb{C})$. Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbf{P}^n(\mathbb{C})$ be the canonical projection. Let now $\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{i=1}^{n+1} \bar{x}_i y_i$, $\mathbf{x} = (x_1, \dots, x_{n+1})$, $\mathbf{y} = (y_1, \dots, y_{n+1}) \in \mathbb{C}^{n+1}$ be the standard scalar product in the space \mathbb{C}^{n+1} .

The metric \tilde{g} of the constant holomorphic sectional curvature on the space $\mathbf{P}^n(\mathbb{C})$ is defined by the same formula (8) as on the space $\mathbf{P}^n(\mathbb{H})$, where now $\mathbf{z} \in \mathbb{C}^{n+1} \setminus \{0\}$, $\xi_i \in T_{\mathbf{z}}\mathbb{C}^{n+1}$, $\zeta_i = \pi_* \xi_i \in T_{\pi(\mathbf{z})}(\mathbf{P}^n(\mathbb{C}))$, $i = 1, 2$.

The Riemannian metric g on the space $\mathbf{P}^n(\mathbb{C})$ is

$$g = 4R^2 \text{Re } \tilde{g}. \tag{25}$$

If $n = 2$ it is not difficult to verify (as in section 5) that the homeomorphism $\tau : \mathbf{P}^1(\mathbb{C}) \rightarrow \bar{\mathbb{C}} \cong \mathbb{S}^2$, $\tau(z_1, z_2) = z_1(z_2)^{-1} = z \in \bar{\mathbb{C}}$, transforms (25) into the metric

$$g = \frac{4R^2 dz d\bar{z}}{(1 + |z|^2)^2}$$

of the sectional curvature R^{-2} on the sphere \mathbb{S}^2 .

The left action of the group $G = SU(n+1)$ on the space \mathbb{C}^{n+1} conserves the scalar product $\langle \cdot, \cdot \rangle$ and induces the action on the space $\mathbf{P}^n(\mathbb{C})$, conserving metrics \tilde{g} and g .

The stationary subgroup, corresponding to the point of the space $\mathbf{P}^n(\mathbb{C})$ with homogeneous coordinates $(1 : 0 : \dots : 0)$, is the group $U(n) = SU(n)U(1)$, where the factor $SU(n)$ acts in the standard way on the last n coordinates, and the factor $U(1)$ acts by multiplication of the first coordinate by $e^{i\phi}$ and the second one by $e^{-i\phi}$, $\phi \in R \bmod 2\pi$. Thus $\mathbf{P}^n(\mathbb{C}) = SU(n+1)/U(n)$.

Choose a base of the algebra $\mathfrak{su}(n+1)$ in the form

$$\Psi_{kj} = \frac{1}{2}(E_{kj} - E_{jk}) \quad \Upsilon_{kj} = \frac{\mathbf{i}}{2}(E_{kj} + E_{jk}) \quad 1 \leq k < j \leq n+1 \tag{26}$$

$$\Upsilon_k = \frac{\mathbf{i}}{2}(E_{11} - E_{kk}) \quad 2 \leq k \leq n + 1. \tag{27}$$

The commutators for these elements are easily extracted from (12), taking into account that $\Upsilon_k = \frac{1}{2}(\Upsilon_{11} - \Upsilon_{kk})$ using the notation from (11).

8.2. Algebras $\text{Diff}(\mathbf{P}^n(\mathbb{C})_{\mathbb{S}})$ and $\text{Diff}(\mathbf{H}^n(\mathbb{C})_{\mathbb{S}})$

Consider now the space $\mathbf{P}^n(\mathbb{C})_{\mathbb{S}}$. Due to the isomorphism $\mathbf{P}^1(\mathbb{C}) \cong \mathbf{S}^2$ we again assume that $n \geq 2$.

Let $\tilde{\mathbf{z}}_0 = (1, 0, \dots, 0) \in \mathbb{C}^{n+1}$, an element $\xi_0 \in T_{\tilde{\mathbf{z}}_0}\mathbb{C}^{n+1} \cong \mathbb{C}^{n+1}$ has coordinates $(0, 1, 0, \dots, 0)$. Put $\mathbf{z}_0 = \pi\tilde{\mathbf{z}}_0$, $\zeta_0 = \pi_*\xi_0 \in T_{\mathbf{z}_0}\mathbf{P}^n(\mathbb{C})$.

The stationary subgroup K_0 of the group $SU(n + 1)$, corresponding to the point (\mathbf{z}_0, ζ_0) , is generated by the group $K_1 = SU(n - 1)$, acting on the last $(n - 1)$ th coordinates and by the group $K_2 = U(1)$, acting on the homogeneous coordinates of $\mathbf{P}^n(\mathbb{C})$ as

$$(x_1 : \dots : x_{n+1}) \rightarrow (e^{\mathbf{i}\phi}x_1 : e^{\mathbf{i}\phi}x_2 : e^{-2\mathbf{i}\phi}x_3 : x_4 : \dots : x_{n+1}) \tag{28}$$

$\dim_{\mathbb{R}} K_0 = (n - 1)^2$ and we obtain $K_0 \cong U(n - 1)$.

The algebra \mathfrak{k}_0 of the group K_0 is the linear hull of elements (26) as $3 \leq k < j \leq n + 1$ and elements:

$$\Upsilon_j - \Upsilon_3 = \frac{\mathbf{i}}{2}(E_{33} - E_{jj}) \quad 3 < j \leq n + 1 \quad 2\Upsilon_3 - \Upsilon_2 = \frac{\mathbf{i}}{2}(E_{11} + E_{22} - 2E_{33}).$$

Choose the complementary subspace $\tilde{\mathfrak{p}}$ to the subalgebra \mathfrak{k}_0 in the algebra $\mathfrak{g} = \mathfrak{su}(n + 1)$ as the linear hull of elements:

$$\Psi_{1k} \quad \Upsilon_{1k} \quad 2 \leq k \leq n + 1 \quad \Psi_{2k} \quad \Upsilon_{2k} \quad 3 \leq k \leq n + 1 \quad \Upsilon_* = \Upsilon_2. \tag{29}$$

Taking into account relations (12) it is easily obtained that the expansion $\mathfrak{su}(n + 1) = \tilde{\mathfrak{p}} \oplus \mathfrak{k}_0$ is reductive, i.e. $[\tilde{\mathfrak{p}}, \mathfrak{k}_0] \subset \tilde{\mathfrak{p}}$.

We will obtain the particular case of proposition 1 for $q_1 = 2n - 2, q_2 = 1$ setting

$$\begin{aligned} \Lambda = -\Psi_{12} & \quad e_{\lambda, k-2} = \Psi_{1k} & \quad e_{\lambda, n-3+k} = \Upsilon_{1k} & \quad f_{\lambda, k-2} = -\Psi_{2k} \\ f_{\lambda, n-3+k} = -\Upsilon_{2k} & \quad e_{2\lambda, 1} = \Upsilon_{12} & \quad f_{2\lambda, 1} = \Upsilon_* & \quad k = 3, \dots, n + 1. \end{aligned} \tag{30}$$

Now we find the generators of Ad_{K_0} -invariants in $S(\tilde{\mathfrak{p}})$. The expansion $\tilde{\mathfrak{p}} = \mathfrak{a} \oplus \mathfrak{k}_{\lambda} \oplus \mathfrak{k}_{2\lambda} \oplus \mathfrak{p}_{\lambda} \oplus \mathfrak{p}_{2\lambda}$ is invariant w.r.t. the Ad_{K_0} -action. In the spaces $\mathfrak{a}, \mathfrak{p}_{2\lambda}, \mathfrak{k}_{2\lambda}$ the K_0 -action is trivial that gives the invariants $D_0 = \Lambda, D_4 = \Upsilon_{12}, D_5 = \Upsilon_* \in \mu(\mathfrak{p}^{K_0})$. Operators D_4, D_5 are square roots of their analogues from section 4.

From formulae (30) we see that the space $\mathfrak{p}_{\lambda} \cong \mathbb{C}^{n-1}$ consists of matrices of the form

$$\begin{pmatrix} 0 & -a^* & & & \\ a & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & -\bar{a}_1 & \dots & -\bar{a}_{n-1} \\ 0 & 0 & 0 & \dots & 0 \\ a_1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & 0 & 0 & \dots & 0 \end{pmatrix} \quad a_1, \dots, a_{n-1} \in \mathbb{C}.$$

Similarly, the space $\mathfrak{k}_{\lambda} \cong \mathbb{C}^{n-1}$ consists of matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 & & \\ 0 & 0 & -b^* & & \\ 0 & 0 & & & \\ 0 & b & & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -\bar{b}_1 & \dots & -\bar{b}_{n-1} \\ 0 & b_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n-1} & 0 & \dots & 0 \end{pmatrix} \quad b_1, \dots, b_{n-1} \in \mathbb{C}.$$

The action of the group K_1 on the spaces \mathfrak{p}_λ and \mathfrak{k}_λ is equivalent to the standard action of the group $SU(n-1)$ on the space $\mathbb{C}^{n-1} : a \rightarrow Ua, U \in SU(n-1)$, likewise in section 6. It is easy to verify that the action (28) generates the action $a_1 \rightarrow \exp^{-3i\phi} a_1, a_i \rightarrow \exp^{-i\phi} a_i, b_1 \rightarrow \exp^{-3i\phi} b_1, b_i \rightarrow \exp^{-i\phi} b_i, i = 2, \dots, n-1$. Therefore, the K_0 -action on spaces \mathfrak{p}_λ and \mathfrak{k}_λ is equivalent to the standard $U(n-1)$ -action on \mathbb{C}^{n-1} .

This action has one independent real invariant: $\langle \mathbf{z}, \mathbf{z} \rangle, \mathbf{z} \in \mathbb{C}^{n-1}$, and the diagonal action of $U(n-1)$ on the space $\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \cong \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1}$ has four (independent iff $n \geq 3$) real invariants:

$$\langle \mathbf{z}_1, \mathbf{z}_1 \rangle \in \mathbb{R} \quad \langle \mathbf{z}_2, \mathbf{z}_2 \rangle \in \mathbb{R} \quad \langle \mathbf{z}_1, \mathbf{z}_2 \rangle \in \mathbb{C} \cong \mathbb{R}^2 \quad \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{C}^{n-1}. \tag{31}$$

Denote the corresponding elements from $\mu(\tilde{\mathfrak{p}}^{K_0}) \in U(\mathfrak{g})^{K_0}$ in the following way:

$$D_1 = \sum_{k=3}^{n+1} (\Psi_{1k}^2 + \Upsilon_{1k}^2) \quad D_2 = \sum_{k=3}^{n+1} (\Psi_{2k}^2 + \Upsilon_{2k}^2) \tag{32}$$

$$D_3 = -\frac{1}{2} \sum_{k=3}^{n+1} (\{\Psi_{1k}, \Psi_{2k}\} + \{\Upsilon_{1k}, \Upsilon_{2k}\}) \quad \square = \frac{1}{2} \sum_{k=3}^{n+1} (-\{\Psi_{1k}, \Upsilon_{2k}\} + \{\Psi_{2k}, \Upsilon_{1k}\}).$$

In this case only operator \square is new w.r.t. section 4.

If $n = 2$, then there is the unique independent relation between invariants (31):

$$|\langle \mathbf{z}_1, \mathbf{z}_2 \rangle|^2 = |\bar{z}_1 z_2|^2 = |z_1|^2 |z_2|^2 = \langle \mathbf{z}_1, \mathbf{z}_1 \rangle \langle \mathbf{z}_2, \mathbf{z}_2 \rangle \quad \mathbf{z}_1 = z_1 \quad \mathbf{z}_2 = z_2 \in \mathbb{C}. \tag{33}$$

Thus operators D_0, \dots, D_5, \square generate the algebra $\text{Diff}(\mathbf{P}^n(\mathbb{C})_{\mathbb{S}})$.

The degrees of the generators are as follows:

$$\text{deg}(D_0) = \text{deg}(D_4) = \text{deg}(D_5) = 1 \quad \text{deg}(D_1) = \text{deg}(D_2) = \text{deg}(D_3) = \text{deg}(\square) = 2. \tag{34}$$

The operators D_3, D_4 are symmetric and the operators D_0, \square, D_5 are skew symmetric w.r.t. the transposition of coordinates z_1 and z_2 . The operators D_1 and D_2 turn into each other under this transposition.

In order to get the generators of the algebra $\text{Diff}(\mathbf{H}^n(\mathbb{C})_{\mathbb{S}})$ we can use the formal substitution:

$$\begin{aligned} \Lambda &\rightarrow \mathbf{i}\Lambda & \Psi_{1k} &\rightarrow \mathbf{i}\Psi_{1k} & \Upsilon_{1k} &\rightarrow \mathbf{i}\Upsilon_{1k} & \Upsilon_{12} &\rightarrow \mathbf{i}\Upsilon_{12} \\ \Psi_{2k} &\rightarrow \Psi_{2k} & \Upsilon_{2k} &\rightarrow \Upsilon_{2k} & \Upsilon_* &\rightarrow \Upsilon_* & k &= 3, \dots, n+1. \end{aligned}$$

This substitution produces the following substitution for the generators D_0, \dots, D_5, \square :

$$\begin{aligned} D_0 &\rightarrow \mathbf{i}\bar{D}_0 & D_1 &\rightarrow -\bar{D}_1 & D_2 &\rightarrow \bar{D}_2 \\ D_3 &\rightarrow \mathbf{i}\bar{D}_3 & D_4 &\rightarrow \mathbf{i}\bar{D}_4 & \square &\rightarrow \mathbf{i}\bar{\square} & D_5 &\rightarrow \bar{D}_5. \end{aligned} \tag{35}$$

The operators $\bar{D}_0, \dots, \bar{D}_5, \bar{\square}$ generate the algebra $\text{Diff}(\mathbf{H}^n(\mathbb{C})_{\mathbb{S}})$.

8.3. Relations in algebras $\text{Diff}(\mathbf{P}^n(\mathbb{C})_{\mathbb{S}})$ and $\text{Diff}(\mathbf{H}^n(\mathbb{C})_{\mathbb{S}})$

The commutative relation for the algebra $\text{Diff}(\mathbf{P}^n(\mathbb{C})_{\mathbb{S}})$ are as follows:

$$\begin{aligned} [D_0, D_1] &= -D_3 & [D_0, D_2] &= D_3 & [D_0, D_3] &= \frac{1}{2}(D_1 - D_2) \\ [D_0, D_4] &= -D_5 & [D_0, D_5] &= D_4 & [D_0, \square] &= 0 \\ [D_1, D_2] &= -\{D_0, D_3\} - \{\square, D_4\} & [D_1, D_3] &= -\frac{1}{2}\{D_0, D_1\} + \frac{1}{2}\{\square, D_5\} + \frac{(n-1)^2}{4}D_0 \\ [D_1, D_4] &= \square & [D_1, D_5] &= 0 & [D_1, \square] &= -\frac{1}{2}\{D_1, D_4\} - \frac{1}{2}\{D_3, D_5\} + \frac{(n-1)^2}{4}D_4 \end{aligned}$$

$$\begin{aligned}
[D_2, D_3] &= \frac{1}{2}\{D_0, D_2\} + \frac{1}{2}\{\square, D_5\} - \frac{(n-1)^2}{4}D_0 & [D_1, D_4] &= \square \\
[D_2, D_5] &= 0 & [D_2, \square] &= \frac{1}{2}\{D_2, D_4\} - \frac{1}{2}\{D_3, D_5\} - \frac{(n-1)^2}{4}D_4 & [D_3, D_4] &= 0 \\
[D_3, D_5] &= \square & [D_3, \square] &= -\frac{1}{4}\{D_1 + D_2, D_5\} + \frac{(n-1)^2}{4}D_5 & [D_4, D_5] &= -D_0 \\
[D_4, \square] &= \frac{1}{2}(D_1 - D_2) & [D_5, \square] &= D_3.
\end{aligned}$$

If $n > 2$ then there are no relations of the second type. If $n = 2$ then there is one relation of the second type due to (33):

$$\frac{1}{2}\{D_1, D_2\} - D_3^2 - \square^2 - \frac{1}{4}(D_0^2 + D_4^2 + D_5^2) = 0. \quad (36)$$

It is easy to verify that the operator $D^* = D_0^2 + D_1 + D_2 + D_4^2 + D_5^2$ lies in the centre of the algebra $\text{Diff}(\mathbf{P}^n(\mathbb{C})_{\mathbb{S}})$ in accordance with section 4.

Using substitution (35) we obtain analogous relations for the algebra $\text{Diff}(\mathbf{H}^n(\mathbb{C})_{\mathbb{S}})$.

The commutative relation are now as follows:

$$\begin{aligned}
[\bar{D}_0, \bar{D}_1] &= \bar{D}_3 & [\bar{D}_0, \bar{D}_2] &= \bar{D}_3 & [\bar{D}_0, \bar{D}_3] &= \frac{1}{2}(\bar{D}_2 + \bar{D}_1) & [\bar{D}_0, \bar{D}_4] &= \bar{D}_5 \\
[\bar{D}_0, \bar{D}_5] &= \bar{D}_4 & [\bar{D}_0, \bar{\square}] &= 0 & [\bar{D}_1, \bar{D}_2] &= -\{\bar{D}_0, \bar{D}_3\} - \{\bar{\square}, \bar{D}_4\} \\
[\bar{D}_1, \bar{D}_3] &= -\frac{1}{2}\{\bar{D}_0, \bar{D}_1\} - \frac{1}{2}\{\bar{\square}, \bar{D}_5\} - \frac{(n-1)^2}{4}\bar{D}_0 & [\bar{D}_1, \bar{D}_4] &= -\bar{\square} \\
[\bar{D}_1, \bar{D}_5] &= 0 & [\bar{D}_1, \bar{\square}] &= -\frac{1}{2}\{\bar{D}_1, \bar{D}_4\} + \frac{1}{2}\{\bar{D}_3, \bar{D}_5\} - \frac{(n-1)^2}{4}\bar{D}_4 \\
[\bar{D}_2, \bar{D}_3] &= \frac{1}{2}\{\bar{D}_0, \bar{D}_2\} + \frac{1}{2}\{\bar{\square}, \bar{D}_5\} - \frac{(n-1)^2}{4}\bar{D}_0 & [\bar{D}_1, \bar{D}_4] &= -\bar{\square} \\
[\bar{D}_2, \bar{D}_5] &= 0 & [\bar{D}_2, \bar{\square}] &= \frac{1}{2}\{\bar{D}_2, \bar{D}_4\} - \frac{1}{2}\{\bar{D}_3, \bar{D}_5\} - \frac{(n-1)^2}{4}\bar{D}_4 \\
[\bar{D}_3, \bar{D}_4] &= 0 & [\bar{D}_3, \bar{D}_5] &= \bar{\square} & [\bar{D}_3, \bar{\square}] &= -\frac{1}{4}\{\bar{D}_1 - \bar{D}_2, \bar{D}_5\} - \frac{(n-1)^2}{4}\bar{D}_5 \\
[\bar{D}_4, \bar{D}_5] &= -\bar{D}_0 & [\bar{D}_4, \bar{\square}] &= \frac{1}{2}(\bar{D}_1 + \bar{D}_2) & [\bar{D}_5, \bar{\square}] &= \bar{D}_3.
\end{aligned}$$

If $n > 2$ then there are no relations of the second type. If $n = 2$ then there is one relation of the second type analogous to (36):

$$\frac{1}{2}\{\bar{D}_1, \bar{D}_2\} - \bar{D}_3^2 - \bar{\square}^2 - \frac{1}{4}(\bar{D}_0^2 + \bar{D}_4^2 - \bar{D}_5^2) = 0. \quad (37)$$

The operator $\bar{D}^* = \bar{D}_0^2 + \bar{D}_1 - \bar{D}_2 + \bar{D}_4^2 - \bar{D}_5^2$ lies in the centre of the algebra $\text{Diff}(\mathbf{H}^n(\mathbb{C})_{\mathbb{S}})$.

9. Algebras $\text{Diff}(\mathbf{P}^n(\mathbb{R})_{\mathbb{S}})$, $\text{Diff}(\mathbf{S}_{\mathbb{S}}^n)$ and $\text{Diff}(\mathbf{H}^n(\mathbb{R})_{\mathbb{S}})$

9.1. Generators of algebras $\text{Diff}(\mathbf{S}_{\mathbb{S}}^n)$ and $\text{Diff}(\mathbf{H}^n(\mathbb{R})_{\mathbb{S}})$

Let now $\langle \cdot, \cdot \rangle$ be the standard scalar product in the space \mathbb{R}^{n+1} . The equation $\langle \mathbf{x}, \mathbf{x} \rangle = R > 0$ defines the sphere $\mathbf{S}^n \cong SO(n+1)/SO(n) \subset \mathbb{R}^{n+1}$ of the radius R with the induced metric on it. The space $\mathbf{P}^n(\mathbb{R})$ is the factor space of \mathbf{S}^n w.r.t. the relation: $\mathbf{x} \sim -\mathbf{x}$. Below we will show that algebras $\text{Diff}(\mathbf{P}^n(\mathbb{R})_{\mathbb{S}})$ and $\text{Diff}(\mathbf{S}_{\mathbb{S}}^n)$ are isomorphic.

The spaces $\mathbf{S}_{\mathbb{S}}^1, \mathbf{P}^1(\mathbb{R})_{\mathbb{S}}$ are one dimensional and the algebra of invariant differential operators on them is generated by one differential operator of the first order. Therefore, we again suppose that $n \geq 2$.

Let

$$\Psi_{kj} = \frac{1}{2}(E_{kj} - E_{jk}) \quad 1 \leq k < j \leq n+1 \quad (38)$$

be the base of the algebra $\mathfrak{so}(n+1)$. The commutative relations for them are contained in (12).

Consider the space \mathbf{S}_S^n . Let $\tilde{\mathbf{z}}_0 = (1, 0, \dots, 0) \in \mathbb{R}^{n+1}$, an element $\xi_0 \in T_{\tilde{\mathbf{z}}_0}\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$ has coordinates $(0, 1, 0, \dots, 0)$. Put $\mathbf{z}_0 = \pi\tilde{\mathbf{z}}_0$, $\zeta_0 = \pi_*\xi_0 \in T_{\mathbf{z}_0}\mathbf{S}_S^n$.

The stationary subgroup K_0 of the group $SO(n+1)$, corresponding to the point $(\mathbf{z}_0, \zeta_0) \in \mathbf{S}_S^n$, is the group $SO(n-1)$, acting onto the last $(n-1)$ th coordinates.

The group $SO(n+1)$ is a group covering of the identity component G of the isometry group for $\mathbf{P}^n(\mathbb{R})$. The group $K_0 = SO(n-1) \subset SO(n+1)$ is a group covering of its analogue $K'_0 \subset G$. The kernel of such covering is a normal subgroup of $SO(n-1)$ that lies in the centre of $SO(n-1)$ [17] (lecture 9). Therefore, the orbits of Ad_{K_0} - and $\text{Ad}_{K'_0}$ -actions on $\mathfrak{p} \subset \mathfrak{g}$ coincide and the construction from section 2 implies the isomorphism $\text{Diff}(\mathbf{P}^n(\mathbb{R})_S) \cong \text{Diff}(\mathbf{S}_S^n)$.

The algebra \mathfrak{k}_0 of the group K_0 is the linear hull of elements Ψ_{kj} as $3 \leq k < j \leq n+1$. Choose the complementary subspace $\tilde{\mathfrak{p}}$ to the subalgebra \mathfrak{k}_0 in the algebra $\mathfrak{g} = \mathfrak{so}(n+1)$ as the linear hull of elements:

$$\Psi_{1k} \quad 2 \leq k \leq n+1 \quad \Psi_{2k} \quad 3 \leq k \leq n+1. \quad (39)$$

The expansion $\mathfrak{so}(n+1) = \tilde{\mathfrak{p}} \oplus \mathfrak{so}(n-1)$ is reductive.

We will obtain the particular case of proposition 1 for $q_1 = 0$, $q_2 = n-1$ setting

$$\Lambda = -2\Psi_{12} \quad e_{2\lambda, k-2} = 2\Psi_{1k} \quad f_{2\lambda, k-2} = -2\Psi_{2k} \quad k = 3, \dots, n+1. \quad (40)$$

Now we have the expansion $\tilde{\mathfrak{p}} = \mathfrak{a} \oplus \mathfrak{k}_{2\lambda} \oplus \mathfrak{p}_{2\lambda}$, which is invariant w.r.t. the Ad_{K_0} -action. It is easy to see that on the space \mathfrak{a} the K_0 -action is trivial and on the spaces $\mathfrak{k}_{2\lambda}$ and $\mathfrak{p}_{2\lambda}$ it is equivalent to the standard action of the group $SO(n-1)$ on the space \mathbb{R}^{n-1} . The K_0 -action on the space \mathfrak{a} has the invariant $D_0 = \Lambda$. The description of base K_0 -invariants on the space $\mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda}$ is different in cases $n = 2$, $n = 3$ and $n \geq 4$.

9.1.1. The case $n \geq 4$. The $SO(n-1)$ -action in \mathbb{R}^{n-1} has one independent real invariant: $\langle \mathbf{z}, \mathbf{z} \rangle$, $\mathbf{z} \in \mathbb{R}^{n-1}$, and the diagonal action of $SO(n-1)$ on the space $\mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda} \cong \mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1}$ has three independent real invariants:

$$\langle \mathbf{z}_1, \mathbf{z}_1 \rangle \quad \langle \mathbf{z}_2, \mathbf{z}_2 \rangle \quad \langle \mathbf{z}_1, \mathbf{z}_2 \rangle \quad \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^{n-1}. \quad (41)$$

Denote the corresponding elements from $\mu(\tilde{\mathfrak{p}}^{K_0}) \in U(\mathfrak{g})^{K_0}$ in the following way:

$$D_1 = 4 \sum_{k=3}^{n+1} \Psi_{1k}^2 \quad D_2 = 4 \sum_{k=3}^{n+1} \Psi_{2k}^2 \quad D_3 = -2 \sum_{k=3}^{n+1} \{\Psi_{1k}, \Psi_{2k}\}.$$

All these invariants were found in section 4 for the general situation.

Thus operators D_0, D_1, D_2, D_3 generate the algebra $\text{Diff}(\mathbf{S}_S^n)$.

The degrees of the generators are as follows:

$$\deg(D_0) = 1 \quad \deg(D_1) = \deg(D_2) = \deg(D_3) = 2. \quad (42)$$

The operator D_3 is symmetric and the operators D_0 is skew symmetric w.r.t. the transposition of coordinates z_1 and z_2 . The operators D_1 and D_2 turn into each other under this transposition.

9.1.2. The case $n = 2$. In this case K_0 is the trivial group and the independent invariants are $D_0, D_1 = e_{2\lambda, 1}, D_2 = f_{2\lambda, 1}$. Thus the algebra $\text{Diff}(\mathbf{S}_S^2)$ is isomorphic to $U(\mathfrak{so}(3))$. The centre of this algebra is generated by the operator $D_0^2 + D_1^2 + D_2^2$.

9.1.3. *The case $n = 3$.* In this case $K_0 = \mathfrak{so}(2)$ and we have the additional (with respect to the case $n \geq 4$) invariant of the second order

$$\square = 2(\{\Psi_{13}, \Psi_{24}\} - \{\Psi_{14}, \Psi_{23}\}).$$

It is algebraically connected with operators D_0, D_1, D_2, D_3 which are defined as in the case $n \geq 4$.

9.1.4. *Generators of the algebra $\text{Diff}(\mathbf{H}^n(\mathbb{R})_{\mathbb{S}})$.* First, let $n \geq 4$. In order to get the generators of the algebra $\text{Diff}(\mathbf{H}^n(\mathbb{R})_{\mathbb{S}})$ we can use the formal substitution:

$$\Lambda \rightarrow \mathbf{i}\Lambda \quad \Psi_{1k} \rightarrow \mathbf{i}\Psi_{1k} \quad \Psi_{2k} \rightarrow \Psi_{2k} \quad k = 3, \dots, n+1.$$

This substitution produces the following substitution for the generators D_0, \dots, D_3 :

$$D_0 \rightarrow \mathbf{i}\bar{D}_0 \quad D_1 \rightarrow -\bar{D}_1 \quad D_2 \rightarrow \bar{D}_2 \quad D_3 \rightarrow \mathbf{i}\bar{D}_3. \quad (43)$$

The operators $\bar{D}_0, \dots, \bar{D}_3$ generate the algebra $\text{Diff}(\mathbf{H}^n(\mathbb{R})_{\mathbb{S}})$.

In the case $n = 3$ we have the additional substitution $\square \rightarrow \mathbf{i}\bar{\square}$ and the operators $\bar{D}_0, \dots, \bar{D}_3, \bar{\square}$ generate the algebra $\text{Diff}(\mathbf{H}^3(\mathbb{R})_{\mathbb{S}})$.

In the case $n = 2$ we obtain the substitution

$$D_0 \rightarrow \mathbf{i}\bar{D}_0 \quad D_1 \rightarrow \mathbf{i}\bar{D}_1 \quad D_2 \rightarrow \bar{D}_2.$$

The algebra $\text{Diff}(\mathbf{H}^2(\mathbb{R})_{\mathbb{S}})$ is isomorphic to $U(\mathfrak{so}(2, 1))$ and its centre is generated by the operator $D_0^2 + D_1^2 - D_2^2$.

9.2. Relations in algebras $\text{Diff}(\mathbf{S}_{\mathbb{S}}^n)$ and $\text{Diff}(\mathbf{H}^n(\mathbb{R})_{\mathbb{S}})$

Here we shall consider only the case $n \geq 3$, since $\text{Diff}(\mathbf{S}_{\mathbb{S}}^2) \cong U(\mathfrak{so}(3))$ and $\text{Diff}(\mathbf{H}^2(\mathbb{R})_{\mathbb{S}}) \cong U(\mathfrak{so}(2, 1))$.

The commutative relations for the algebra $\text{Diff}(\mathbf{S}_{\mathbb{S}}^n)$ are as follows:

$$\begin{aligned} [D_0, D_1] &= -2D_3 & [D_0, D_2] &= 2D_3 & [D_0, D_3] &= D_1 - D_2 \\ [D_1, D_2] &= -2\{D_0, D_3\} & [D_1, D_3] &= -\{D_0, D_1\} + \frac{(n-1)(n-3)}{2}D_0 \\ [D_2, D_3] &= \{D_0, D_2\} - \frac{(n-1)(n-3)}{2}D_0. \end{aligned}$$

For $n = 3$ the additional operator \square lies in the centre of the algebra $\text{Diff}(\mathbf{S}_{\mathbb{S}}^3)$.

If $n > 3$ then there are no relations of the second type. If $n = 3$ then there is one relation of the second type:

$$\frac{1}{2}\{D_1, D_2\} - D_0^2 = D_3^2 + \square^2. \quad (44)$$

It is easy to verify that the operators $D_1^* = D_0^2 + D_1 + D_2$ and

$$D_2^* = \frac{1}{2}\{D_1, D_2\} - D_3^2 + \left(1 - \frac{(n-3)(n-1)}{4}\right)(D_1 + D_2)$$

lie in the centre of the algebra $\text{Diff}(\mathbf{S}_{\mathbb{S}}^n)$. If $n = 3$ it holds that $D_2^* = \square^2 + D_1^*$ due to (44). In this case two operators D_1^* and \square generate the centre of the algebra $\text{Diff}(\mathbf{S}_{\mathbb{S}}^3)$.

Using substitution (43) we obtain analogous relations for the algebra $\text{Diff}(\mathbf{H}^n(\mathbb{R})_{\mathbb{S}})$.

The commutative relation are now as follows:

$$\begin{aligned} [\bar{D}_0, \bar{D}_1] &= 2\bar{D}_3 & [\bar{D}_0, \bar{D}_2] &= 2\bar{D}_3 & [\bar{D}_0, \bar{D}_3] &= \bar{D}_2 + \bar{D}_1 \\ [\bar{D}_1, \bar{D}_2] &= -2\{\bar{D}_0, \bar{D}_3\} & [\bar{D}_1, \bar{D}_3] &= -\{\bar{D}_0, \bar{D}_1\} - \frac{(n-1)(n-3)}{2}\bar{D}_0 \\ [\bar{D}_2, \bar{D}_3] &= \{\bar{D}_0, \bar{D}_2\} - \frac{(n-1)(n-3)}{2}\bar{D}_0 \end{aligned}$$

and for $n = 3$ also

$$[\bar{D}_0, \square] = [\bar{D}_1, \square] = [\bar{D}_2, \square] = [\bar{D}_3, \square] = 0.$$

The first three relations were found in [4], but the other relations were not calculated there.

If $n > 3$ then there are no relations of the second type. If $n = 3$ then there is one relation of the second type analogous to (44):

$$\frac{1}{2}\{\bar{D}_1, \bar{D}_2\} - \bar{D}_0^2 = \bar{D}_3^2 + \square^2. \tag{45}$$

The operators $\bar{D}_1^* = \bar{D}_0^2 + \bar{D}_1 - \bar{D}_2$ and

$$\bar{D}_2^* = \frac{1}{2}\{\bar{D}_1, \bar{D}_2\} - \bar{D}_3^2 + \left(1 - \frac{(n-3)(n-1)}{4}\right)(\bar{D}_1 - \bar{D}_2)$$

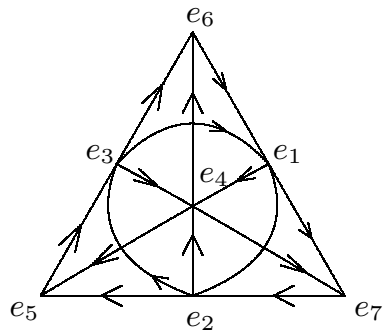
lie in the centre of the algebra $\text{Diff}(\mathbf{H}^n(\mathbb{R})_{\mathbb{S}})$ and if $n = 3$ it holds that $\bar{D}_2^* = \square^2 + \bar{D}_1^*$ due to (45). In this case the operators \bar{D}_1^* and \square generate the centre of the algebra $\text{Diff}(\mathbf{H}_{\mathbb{S}}^3)$.

10. The model of the space $\mathbf{P}^2(\mathbb{C}a)$

Our description of Caley algebra $\mathbb{C}a$ and the octonionic projective plane $\mathbf{P}^2(\mathbb{C}a)$ in this section is based on [16–18].

10.1. The algebra $\mathbb{C}a$

According to Frobenius theorem there are only four finite-dimensional division algebras over \mathbb{R} : \mathbb{R} itself and algebras $\mathbb{C}, \mathbb{H}, \mathbb{C}a$. The latter is an eight-dimensional normed division algebra of octonions. It is noncommutative and nonassociative, but alternative, i.e. for any two elements $\xi, \eta \in \mathbb{C}a$ it holds that $(\xi\eta)\eta = \xi(\eta\eta)$ and $\xi(\xi\eta) = (\xi\xi)\eta$. The group of all automorphisms of $\mathbb{C}a$ is the exceptional simple compact 14-dimensional Lie group G_2 . The standard base of $\mathbb{C}a$ over \mathbb{R} is $\{e_i\}_{i=0}^7$, where $e_0 = 1 \in \mathbb{R}$ and $e_i^2 = -1, e_i e_j = -e_j e_i, i, j = 1, \dots, 7, i \neq j$. The elements $\{e_i\}_{i=1}^7$ are multiplied according to the following scheme:



Here $e_i e_j = e_k$ if these elements lie on one line or on the circle and are ordered by arrows as e_i, e_j, e_k . The conjugation $\iota : \mathbb{C}a \mapsto \mathbb{C}a$ acts as $\iota(e_0) \equiv \bar{e}_0 = e_0, \iota(e_i) \equiv \bar{e}_i = -e_i, i = 1, \dots, 7$ and is extended by linearity over whole $\mathbb{C}a$. Let $\text{Re } \xi = \frac{1}{2}(\xi + \bar{\xi}), \text{Im } \xi = \frac{1}{2}(\xi - \bar{\xi}), \xi \in \mathbb{C}a$. Define the scalar product in $\mathbb{C}a$ by the formula: $\langle \eta, \xi \rangle = \frac{1}{2}(\bar{\eta}\xi + \bar{\xi}\eta) = \text{Re}(\bar{\xi}\eta) = \text{Re}(\bar{\eta}\xi) \in \mathbb{R}$ and the norm by the formula $\|\eta\| = \langle \eta, \eta \rangle^{1/2}$. In the algebra $\mathbb{C}a$ every two elements generate an associative subalgebra and the following central Moufang identity is valid:

$$u \cdot xy \cdot u = ux \cdot yu \quad u, x, y \in \mathbb{C}a. \tag{46}$$

Here we use the notation $u \cdot xy := u(xy), xy \cdot u := (xy)u$.

There are the following descriptions of spinor and vector representations (all eight dimensional) of the group $\text{Spin}(8)$ in \mathbb{Ca} [17, 19], which will be used later. Define linear operators in \mathbb{Ca} :

$$\begin{aligned} L_\alpha &: \xi \mapsto \frac{1}{2}e_\alpha \xi & \alpha = 1, \dots, 7 \quad \xi \in \mathbb{Ca} \\ L_{\alpha,\beta} &: \xi \mapsto \frac{1}{2}e_\alpha(e_\beta \xi) & 1 \leq \alpha < \beta \leq 7 \quad \xi \in \mathbb{Ca}. \end{aligned}$$

These operators are generators of the left spinor representation of the group $\text{Spin}(8)$, i.e. they are the images of some base of the Lie algebra $\text{spin}(8)$ under this representation. Similarly, operators

$$\begin{aligned} R_\alpha &: \xi \mapsto \frac{1}{2}\xi e_\alpha & \alpha = 1, \dots, 7 \quad \xi \in \mathbb{Ca}, \\ R_{\alpha,\beta} &: \xi \mapsto \frac{1}{2}(\xi e_\beta)e_\alpha & 1 \leq \alpha < \beta \leq 7 \quad \xi \in \mathbb{Ca} \end{aligned}$$

are generators of the right spinor representation of the group $\text{Spin}(8)$. All these operators are skew symmetric w.r.t. the scalar product in \mathbb{Ca} .

The formulae above define operators $L_{\alpha,\beta}, R_{\alpha,\beta}$ also for $1 \leq \beta < \alpha \leq 7$. If \mathbb{Ca}' is the space of pure imaginary octonions, $u \in \mathbb{Ca}', \xi \in \mathbb{Ca}$, then due to the alternativity of \mathbb{Ca} :

$$\xi u \cdot u = \xi u^2 = -\xi |u|^2 = -|u|^2 \xi = u \cdot u \xi.$$

For $u = e_\alpha + e_\beta, 1 \leq \alpha < \beta \leq 7$ it holds that

$$\begin{aligned} -2\xi &= -\xi |e_\alpha + e_\beta|^2 = \xi(e_\alpha + e_\beta) \cdot (e_\alpha + e_\beta) = \xi e_\alpha \cdot e_\alpha + \xi e_\alpha \cdot e_\beta + \xi e_\beta \cdot e_\alpha + \xi e_\beta \cdot e_\beta \\ &= -\xi + \xi e_\alpha \cdot e_\beta + \xi e_\beta \cdot e_\alpha - \xi \end{aligned}$$

and $\xi e_\alpha \cdot e_\beta + \xi e_\beta \cdot e_\alpha = 0$. Similarly, $e_\alpha \cdot e_\beta \xi + e_\beta \cdot e_\alpha \xi = 0$. For $0 \leq i, j \leq 7, i \neq j$ we can write more general formulae, useful in the following:

$$e_i \cdot e_j \xi = -\bar{e}_j \cdot \bar{e}_i \xi \quad \xi e_i \cdot e_j = -\xi \bar{e}_j \cdot \bar{e}_i \quad \xi \in \mathbb{Ca}. \quad (47)$$

In particular, we have $L_{\alpha,\beta} = -L_{\beta,\alpha}, R_{\alpha,\beta} = -R_{\beta,\alpha}, 1 \leq \alpha, \beta \leq 7, \alpha \neq \beta$.

For the element $g \in \text{Spin}(8)$ denote by g^L, g^R and g^V its images under left spinor, right spinor and vector representation, respectively. The following proposition is a version of the *triality principle* for the group $\text{Spin}(8)$.³

Proposition 2 ([17]). *For any element $g \in \text{Spin}(8)$ it holds that*

$$g^V(\xi \eta) = g^L(\xi) \cdot g^R(\eta) \quad \xi, \eta \in \mathbb{Ca}. \quad (48)$$

Conversely, if A, B, C are orthogonal operators $\mathbb{Ca} \mapsto \mathbb{Ca}$ such that

$$A(\xi \eta) = B(\xi) \cdot C(\eta)$$

for any $\xi, \eta \in \mathbb{Ca}$, then there exists unique $g \in \text{Spin}(8)$ such that $A = g^V, B = g^L, C = g^R$.

From equation (48) we obtain its infinitesimal analogues:

$$V_i(\xi \eta) = L_i(\xi) \cdot \eta + \xi \cdot R_i(\eta) \quad i = 1, \dots, 7 \quad (49)$$

$$V_{i,j}(\xi \eta) = L_{i,j}(\xi) \cdot \eta + \xi \cdot R_{i,j}(\eta) \quad 1 \leq i < j \leq 7 \quad \xi, \eta \in \mathbb{Ca} \quad (50)$$

where V_i and $V_{i,j}$ are generators of the vector representation of the group $\text{Spin}(8)$.

³ Other versions of this principle are in [16].

10.2. The Jordan algebra $\mathfrak{h}_3(\mathbb{C}a)$

The *Hermitian conjugation* $A \mapsto A^*$ for a square matrix with octonion entries is defined as the composition of octonionic conjugation and transposition of A , similar to complex or quaternion cases. A matrix A is called *Hermitian* iff $A^* = A$. The *simple exceptional Jordan algebra* $\mathfrak{h}_3(\mathbb{C}a)$ consists of all Hermitian 3×3 matrices with octonion entries. It is endowed with the Jordan commutative multiplication:

$$X \circ Y = \frac{1}{2}(XY + YX) \quad X, Y \in \mathfrak{h}_3(\mathbb{C}a).$$

This multiplication satisfies the identity $(X^2 \circ Y) \circ X = X^2 \circ (Y \circ X)$ which is the condition for an algebra with commutative (but not necessarily associative) multiplication to be Jordan. The Jordan algebra $\mathfrak{h}_3(\mathbb{C}a)$ is 27 dimensional over \mathbb{R} . Its every element can be represented in the form

$$X = a_1 E_1 + a_2 E_2 + a_3 E_3 + X_1(\xi_1) + X_2(\xi_2) + X_3(\xi_3) \quad (51)$$

where

$$\begin{aligned} E_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & E_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & E_3 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ X_1(\xi) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \xi \\ 0 & \bar{\xi} & 0 \end{pmatrix} & X_2(\xi) &= \begin{pmatrix} 0 & 0 & \bar{\xi} \\ 0 & 0 & 0 \\ \xi & 0 & 0 \end{pmatrix} & X_3(\xi) &= \begin{pmatrix} 0 & \xi & 0 \\ \bar{\xi} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

$a_i \in \mathbb{R}$, $\xi_i \in \mathbb{C}a$, $i = 1, 2, 3$. It is easy to show that

$$\begin{aligned} E_i \circ E_j &= \begin{cases} E_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \\ E_i \circ X_j(\xi) &= \begin{cases} 0 & \text{if } i = j \\ \frac{1}{2} X_j(\xi) & \text{if } i \neq j \end{cases} \\ X_i(\xi) \circ X_j(\eta) &= \begin{cases} (\xi, \eta)(E - E_i) & \text{if } i = j \\ \frac{1}{2} X_{i+j}(\xi\eta) & \text{if } j \equiv i + 1 \pmod{3} \end{cases} \end{aligned} \quad (52)$$

where $E = E_1 + E_2 + E_3$ is the unit matrix. In the last formula all indices are considered modulo 3.

The group of all automorphisms of the Jordan algebra $\mathfrak{h}_3(\mathbb{C}a)$ is the exceptional simple compact 52-dimensional real Lie group F_4 . This group conserves the following bilinear and trilinear functionals: $\mathcal{A}(X, Y) = \text{Tr}(X \circ Y)$, $\mathcal{B}(X, Y, Z) = \mathcal{A}(X \circ Y, Z)$. Conversely, every linear operator $\mathfrak{h}_3(\mathbb{C}a) \mapsto \mathfrak{h}_3(\mathbb{C}a)$, conserving these two functionals, lies in F_4 .

Define the norm of the element (51) as $\|X\|^2 = \mathcal{A}(X, X) = \sum_{i=1}^3 (a_i^2 + 2|\xi_i|^2)$. The last equality is the consequence of (52).

Theorem 2 (Freudenthal). *For any $X \in \mathfrak{h}_3(\mathbb{C}a)$ there exists an automorphism $\Phi \in F_4$, such that*

$$\Phi X = \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3 \quad (53)$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3$, and the form (53) is uniquely determined by X . Two elements from $\mathfrak{h}_3(\mathbb{C}a)$ lie on the same orbit of F_4 iff their diagonal forms (53) are the same.

10.3. The octonionic projective plane $\mathbf{P}^2(\mathbb{C}a)$

Elements $X \in \mathfrak{h}_3(\mathbb{C}a)$ satisfying conditions

$$X^2 = X \quad \text{Tr } X = 1 \quad (54)$$

form the octonionic projective plane $\mathbf{P}^2(\mathbb{C}a)$, which is a 16-dimensional real manifold. Automorphisms of $\mathfrak{h}_3(\mathbb{C}a)$ conserve equations (54) and the group F_4 acts on $\mathbf{P}^2(\mathbb{C}a)$. From the Freudenthal theorem and equations (54) it follows that every element of $\mathbf{P}^2(\mathbb{C}a)$ can be transformed by an appropriate element of F_4 to the element E_1 . Thus $\mathbf{P}^2(\mathbb{C}a)$ is a homogeneous space of the group F_4 and calculations in [17] (lecture 16) shows that the stationary subgroup of every point $X \in \mathbf{P}^2(\mathbb{C}a)$ is isomorphic to the group $\text{Spin}(9)$.

Let

$$X = (1 + a_1)E_1 + a_2E_2 + a_3E_3 + X_1(\xi_1) + X_2(\xi_2) + X_3(\xi_3) \in \mathbf{P}^2(\mathbb{C}a)$$

where $a_i, |\xi_i|, i = 1, 2, 3$ are tending to zero. Then due to (52) we have

$$X \circ X = (1 + 2a_1)E_1 + X_2(\xi_2) + X_3(\xi_3) + o\left(\sum_{i=1}^3 (a_i^2 + |\xi_i|^2)\right)$$

and the equality $X \circ X = X$ implies $a_1 = a_2 = a_3 = 0, \xi_1 = 0$. It means that we can identify the tangent space $T_{E_1}\mathbf{P}^2(\mathbb{C}a)$ with the set $\{X_2(\xi_2) + X_3(\xi_3) | \xi_1, \xi_2 \in \mathbb{C}a\}$.

Let $K \subset F_4$ be the stationary subgroup corresponding to the point E_1 and acting by automorphisms in the space $T_{E_1}\mathbf{P}^2(\mathbb{C}a) \simeq \{X_2(\xi_2) + X_3(\xi_3) | \xi_1, \xi_2 \in \mathbb{C}a\}$. Let K_0 be the stationary subgroup of K , corresponding to the element $X_3(1) \in T_{E_1}\mathbf{P}^2(\mathbb{C}a)$.

According to section 4 we calculate the K_0 -action on $T_{E_1}\mathbf{P}^2(\mathbb{C}a)$. For any element $X \in \mathfrak{h}_3(\mathbb{C}a)$ let $\text{Ann } X := \{Y \in \mathfrak{h}_3(\mathbb{C}a) | Y \circ X = 0\}$. Being an automorphism of the algebra $\mathfrak{h}_3(\mathbb{C}a)$, an element $\Phi \in K_0$ conserves the space $\text{Ann } X_3(1)$. It follows from (52) that

$$\text{Ann } X_3(1) = \{a(E_1 - E_2) + bE_3 + X_3(\xi) | a, b \in \mathbb{R}, \xi \in \mathbb{C}a'\}.$$

Let $\Phi(E_1 - E_2) = a(E_1 - E_2) + bE_3 + X_3(\xi)$, then we have

$$\begin{aligned} 1 &= \mathcal{A}(E_1 - E_2, E_1) = \mathcal{A}(\Phi(E_1 - E_2), \Phi(E_1)) \\ &= \mathcal{A}(a(E_1 - E_2) + bE_3 + X_3(\xi), E_1) = a. \end{aligned}$$

This implies $\Phi(E_1 - E_2) = E_1 - E_2 + bE_3 + X_3(\xi)$ and the equality $\|E_1 - E_2\| = \|\Phi(E_1 - E_2)\|$ gives $b = 0, \xi = 0$. This means that $\Phi(E_2) = E_2$ and therefore $\Phi(E_3) = \Phi(E - E_1 - E_2) = E - E_1 - E_2 = E_3$. Thus the group K_0 conserves elements E_1, E_2, E_3 .

Let K' be the subgroup of F_4 conserving element E_1, E_2, E_3 . We see that $K_0 \subset K' \subset K$. Since $\text{Ann } E_1 = \{a_2E_2 + a_3E_3 + X_1(\xi), a_1, a_2 \in \mathbb{R}, \xi \in \mathbb{C}a\}$, then the group K' maps $X_1(\xi) \mapsto X_1(\tilde{\xi})$ and similarly $X_i(\xi_i) \mapsto X_i(\tilde{\xi}_i), i = 1, 2, 3$.

Let $\Phi_i : \mathbb{C}a \mapsto \mathbb{C}a, i = 1, 2, 3$ be orthogonal operators such that $\Phi X_i(\xi_i) = X_i(\Phi_i(\xi_i))$ for $\Phi \in K'$. The last formula in (52) implies

$$\begin{aligned} X_3(\Phi_3(\overline{\xi\eta})) &= \Phi(X_3(\overline{\xi\eta})) = 2\Phi(X_1(\xi) \circ X_2(\eta)) = 2\Phi(X_1(\xi)) \circ \Phi(X_2(\eta)) \\ &= 2X_1(\Phi_1(\xi)) \circ X_2(\Phi_2(\eta)) = X_3(\overline{\Phi_1(\xi)\Phi_2(\eta)}). \end{aligned}$$

It gives

$$\Phi_1(\xi)\Phi_2(\eta) = \overline{\Phi_3(\overline{\xi\eta})} \quad (55)$$

for $\Phi \in K', \xi, \eta \in \mathbb{C}a$.

Denote by $\mathbb{C}a_i, i = 1, 2, 3$ the domains for the operators $\Phi_i, i = 1, 2, 3$. Then $T_{E_1}\mathbf{P}^2(\mathbb{C}a) \simeq \mathbb{C}a_2 \oplus \mathbb{C}a_3$.

Formula (55) and proposition 2 imply

Proposition 3. *Operators Φ_1 and Φ_2 are respectively left and right spinor representations of the group $\text{Spin}(8) \simeq K'$ and the composition $\iota \circ \Phi_3 \circ \iota$ is the vector representation of $\text{Spin}(8)$.*

The group $\text{Spin}(8)$ is the universal (double) covering of the group $SO(8)$ and their Lie algebras $\mathfrak{spin}(8)$ and $\mathfrak{so}(8)$ are isomorphic.

Now consider representations of the Lie algebra \mathfrak{k}' of the group K' in $\mathbb{C}a_i$, $i = 1, 2, 3$. All these representations are faithful. For $A \in \mathfrak{k}'$ denote by $A^{(i)}$ the corresponding skew-symmetric operator in $\mathbb{C}a_i$, $i = 1, 2, 3$. From (55) we obtain the following infinitesimal analogue of the triality principle:

$$A^{(1)}(\xi) \cdot \eta + \xi \cdot A^{(2)}(\eta) = \overline{A^{(3)}(\xi\eta)}. \quad (56)$$

From (49) and (50) we obtain that if $A^{(1)} = L_i$ (respectively $A^{(1)} = L_{i,j}$) then $A^{(2)} = R_i$, $A^{(3)} = \iota \circ V_i \circ \iota$ (respectively $A^{(2)} = R_{i,j}$, $A^{(3)} = \iota \circ V_{i,j} \circ \iota$).

Let us identify the algebra \mathfrak{k}' with its vector representation in $\mathbb{C}a_3$, in particular we put $A \equiv A^{(3)}$ for $A \in \mathfrak{k}'$. By \mathfrak{x} denote the inclusion \mathfrak{k}' into the Lie algebra \mathfrak{f}_4 corresponding to the group F_4 .

By definition, the Lie algebra \mathfrak{k}_0 of the group $K_0 \subset K'$ consists of the skew-symmetric operators in $\mathbb{C}a_3$, transforming $1 \in \mathbb{C}a_3$ into 0. The group K_0 is isomorphic to $\text{Spin}(7)$, acting in $\mathbb{C}a_1$ by the left spinor representation, in $\mathbb{C}a_2$ by the right spinor representation (equivalent for $\text{Spin}(7)$ to the left one, see (63)), and in $\mathbb{C}a'_3$ by the vector representation, which are restrictions of analogous representations of $K' \simeq \text{Spin}(8)$.

Let \mathfrak{m} be the space of 3×3 semi-Hermitian matrices with octonion entries and the zero trace. Let

$$\begin{aligned} Y_1(\xi) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \xi \\ 0 & -\bar{\xi} & 0 \end{pmatrix} & Y_2(\xi) &= \begin{pmatrix} 0 & 0 & -\bar{\xi} \\ 0 & 0 & 0 \\ \xi & 0 & 0 \end{pmatrix} \\ Y_3(\xi) &= \begin{pmatrix} 0 & \xi & 0 \\ -\bar{\xi} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \xi &\in \mathbb{C}a \end{aligned}$$

be elements from \mathfrak{m} and the linear subspace $\mathfrak{m}_0 \subset \mathfrak{m}$ consists of elements of the form

$$\sum_{i=1}^3 Y_i(\xi_i) \quad \xi_i \in \mathbb{C}a.$$

From [17] (lecture 16) we can extract the following proposition:

Proposition 4. *For $Y \in \mathfrak{m}$ the linear operator $\text{ad } Y : \mathfrak{h}_3(\mathbb{C}a) \mapsto \mathfrak{h}_3(\mathbb{C}a)$, acting according to the formula $\text{ad } Y(X) = YX - XY$, $X \in \mathfrak{h}_3(\mathbb{C}a)$ is the differentiation of the algebra $\mathfrak{h}_3(\mathbb{C}a)$. Thus the space \mathfrak{m} is contained in \mathfrak{f}_4 . There is the expansion into the direct sum of linear spaces*

$$\mathfrak{f}_4 \simeq \mathfrak{k}' \oplus \mathfrak{m}_0$$

with the following commutator relations:

$$[\mathfrak{x}A, \text{ad } Y_i(\xi)] = \text{ad } Y_i(A^{(i)}\xi) \quad i = 1, 2, 3 \quad (57)$$

$$[\text{ad } Y_i(\xi), \text{ad } Y_j(\eta)] = \begin{cases} \mathfrak{x}C_{i,\xi,\eta} & \text{if } j = i \\ \text{ad } Y_{i+2}(-\bar{\xi}\eta) & \text{if } j = i + 1 \end{cases} \quad (58)$$

where $A \equiv A^{(3)} \in \mathfrak{k}'$, $\xi, \eta \in \mathbb{C}a$, operators $A^{(i)}$ are from (56), the indices in the last equation are considered modulo 3 and semi-Hermitian operators $C_{i,\xi,\eta} : \mathbb{C}a_3 \mapsto \mathbb{C}a_3$, $i = 1, 2, 3$ are given by the following formulae:

$$\begin{aligned} C_{1,\xi,\eta} &: \zeta \mapsto \zeta \xi \cdot \bar{\eta} - \zeta \eta \cdot \bar{\xi} \\ C_{2,\xi,\eta} &: \zeta \mapsto \bar{\eta} \cdot \xi \zeta - \bar{\xi} \cdot \eta \zeta, \zeta \in \mathbb{C}a \\ C_{3,\xi,\eta} &: \zeta \mapsto 4(\xi, \zeta)\eta - 4(\eta, \zeta)\xi. \end{aligned} \quad (59)$$

The action of operators $\kappa C_{i,\xi,\eta}$ on the spaces $\mathbb{C}a_1$ and $\mathbb{C}a_2$ is obtained from (59) by the cyclic permutation of indices:

$$\begin{aligned} \kappa C_{1,\xi,\eta}|_{\mathbb{C}a_1} &: \zeta \mapsto 4(\xi, \zeta)\eta - 4(\eta, \zeta)\xi \\ \kappa C_{2,\xi,\eta}|_{\mathbb{C}a_1} &: \zeta \mapsto \zeta \xi \cdot \bar{\eta} - \zeta \eta \cdot \bar{\xi} \\ \kappa C_{3,\xi,\eta}|_{\mathbb{C}a_1} &: \zeta \mapsto \bar{\eta} \cdot \xi \zeta - \bar{\xi} \cdot \eta \zeta \\ \kappa C_{1,\xi,\eta}|_{\mathbb{C}a_2} &: \zeta \mapsto \bar{\eta} \cdot \xi \zeta - \bar{\xi} \cdot \eta \zeta \\ \kappa C_{2,\xi,\eta}|_{\mathbb{C}a_2} &: \zeta \mapsto 4(\xi, \zeta)\eta - 4(\eta, \zeta)\xi \\ \kappa C_{3,\xi,\eta}|_{\mathbb{C}a_2} &: \zeta \mapsto \zeta \xi \cdot \bar{\eta} - \zeta \eta \cdot \bar{\xi}. \end{aligned} \quad (60)$$

Note that in [17] (lecture 16) analogues of formulae (55), (57) and the last formula (52) contain errors.

11. Generators of algebras $\text{Diff}(\mathbf{P}^2(\mathbb{C}a)_{\mathbb{S}})$ and $\text{Diff}(\mathbf{H}^2(\mathbb{C}a)_{\mathbb{S}})$

Now we specify the construction from section 4 for the space $M = \mathbf{P}^2(\mathbb{C}a)_{\mathbb{S}}$.

11.1. The special base in $\mathfrak{a} \oplus \mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda}$

It is easily seen that

$$[Y_1(\xi), E_1] = 0 \quad [Y_2(\xi), E_1] = X_2(\xi) \quad [Y_3(\xi), E_1] = -X_3(\xi) \quad \xi \in \mathbb{C}a$$

so we can identify the space $T_{E_1}\mathbf{P}^2(\mathbb{C}a)$ with the space $\{Y_2(\xi) + Y_3(\eta)|\xi, \eta \in \mathbb{C}a\} \subset \mathfrak{m}_0$. From (57) we obtain that the expansion

$$\{Y_2(\xi) + Y_3(\eta)|\xi, \eta \in \mathbb{C}a\} = \{Y_3(\xi)|\xi \in \mathbb{R}\} \oplus \{Y_2(\xi)|\xi \in \mathbb{C}a\} \oplus \{Y_3(\xi)|\xi \in \mathbb{C}a'\}$$

is Ad_{K_0} -invariant and by comparison with sections 3 and 4 we can put

$$\mathfrak{a} := \{Y_3(\xi)|\xi \in \mathbb{R}\} \quad \mathfrak{p}_\lambda := \{Y_2(\xi)|\xi \in \mathbb{C}a\} \quad \mathfrak{p}_{2\lambda} := \{Y_3(\xi)|\xi \in \mathbb{C}a'\}.$$

Let $y = (E_1, \frac{1}{2}X_3(1)) \in \mathbf{P}^2(\mathbb{C}a)_{\mathbb{S}}$, where $\frac{1}{2}X_3(1) \in \mathbb{S}_{E_1}$. We have $T_y\mathbf{P}^2(\mathbb{C}a)_{\mathbb{S}} = T_{E_1}\mathbf{P}^2(\mathbb{C}a) \oplus T_{\frac{1}{2}X_3(1)}\mathbb{S}_{E_1}$ and

$$T_{\frac{1}{2}X_3(1)}\mathbb{S}_{E_1} \simeq \{X_2(\xi)|\xi \in \mathbb{C}a\} \oplus \{X_3(\xi)|\xi \in \mathbb{C}a'\}.$$

Since $\text{ad } Y_1(\xi)(X_3(1)) = -X_2(\xi)$, $\xi \in \mathbb{C}a$, the space $\{X_2(\xi)|\xi \in \mathbb{C}a\} \subset T_{\frac{1}{2}X_3(1)}\mathbb{S}_{E_1}$ is identified with the space $\{Y_1(\xi)|\xi \in \mathbb{C}a\} \subset \mathfrak{m}_0$. Since $\dim_{\mathbb{R}}\{Y_1(\xi)|\xi \in \mathbb{C}a\} = 8 = \dim_{\mathbb{R}}\mathfrak{p}_\lambda$ we denote $\mathfrak{k}_\lambda := \{Y_1(\xi)|\xi \in \mathbb{C}a\}$. Thus $\mathfrak{m}_0 = \mathfrak{a} \oplus \mathfrak{p}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_\lambda$.

Denote by $A_{ij} \in \mathfrak{k}'$, $i \neq j$ the generators of the rotation in the two-dimensional plane, containing elements $e_i, e_j \in \mathbb{C}a_3$, such that $A_{ij}e_j = e_i$, $A_{ij}e_i = -e_j$. The operators A_{ij} , $1 \leq i < j \leq 7$ are the base of the algebra \mathfrak{k}_0 . Similar to the quaternion case the subspace \mathfrak{q} of the algebra \mathfrak{k}' with the base $A_{0\alpha} := A_\alpha$, $\alpha = 1, \dots, 7$ is Ad_{K_0} -invariant and is identified through the K_0 -action on $T_y\mathbf{P}^2(\mathbb{C}a)_{\mathbb{S}}$ with the space $\{X_3(\xi)|\xi \in \mathbb{C}a'\} \subset T_{\frac{1}{2}X_3(1)}\mathbb{S}_{E_1}$. Therefore, we define $\mathfrak{k}_{2\lambda} := \mathfrak{q}$.

Lemma 1. *It holds that*

$$\begin{aligned} A_\alpha^{(1)} = L_\alpha & \quad A_\alpha^{(2)} = R_\alpha & \quad A_{\alpha\beta}^{(1)} = L_{\beta,\alpha} & \quad A_{\alpha\beta}^{(2)} = R_{\beta,\alpha} \\ C_{3,e_\alpha,e_\beta} = 4A_{\beta,\alpha} & \quad C_{3,e_0,e_\alpha} = -4A_\alpha & \quad \alpha, \beta = 1, \dots, 7 & \quad \alpha \neq \beta \end{aligned}$$

Proof. From (56) we have

$$A^{(3)}(\xi) = \overline{A^{(1)}(\bar{\xi})} + \bar{\xi}A^{(2)}(1).$$

Let $A^{(1)} = L_\alpha$, then $A^{(2)} = R_\alpha$ and $A^{(3)}(e_k) = \frac{1}{2}\overline{(e_\alpha\bar{e}_k + \bar{e}_ke_\alpha)} = -\frac{1}{2}(e_k e_\alpha + e_\alpha e_k)$. If $1 \leq k \neq \alpha$, then $e_k e_\alpha = -e_\alpha e_k$ and $A^{(3)}(e_k) = 0$. Therefore, $A^{(3)} = A_\alpha$, since $A^{(3)}(1) = -e_\alpha, A^{(3)}(e_\alpha) = 1$. This proves $A_\alpha^{(1)} = L_\alpha, A_\alpha^{(2)} = R_\alpha$.

Let now $A^{(1)} = L_{\beta,\alpha}$, then $A^{(2)} = R_{\beta,\alpha}$ and $A^{(3)}(e_k) = \frac{1}{2}\overline{(e_\beta \cdot e_\alpha \bar{e}_k + \bar{e}_k \cdot e_\alpha e_\beta)} = \frac{1}{2}(e_k e_\alpha \cdot e_\beta + e_\beta e_\alpha \cdot e_k)$. It is easy to verify by direct computation that if $\alpha = 1, \beta = 2$ then $A^{(3)}(e_k) = 0$, for $k \neq 1, 2$ and $A^{(3)}(e_1) = -e_2, A^{(3)}(e_2) = e_1$. Thus $L_{\beta,\alpha}^{(3)} = A_{12}$. Therefore, $L_{\beta,\alpha}^{(3)} = A_{\alpha\beta}$ for any other pair of e_α, e_β , since the group G_2 of automorphisms of $\mathbb{C}a$ acts transitively on any pair of imaginary units [17] (lecture 15). This proves $A_{\alpha\beta}^{(1)} = L_{\beta,\alpha}, A_{\alpha\beta}^{(2)} = R_{\beta,\alpha}$.

The last two equalities of this lemma are obvious. □

Let us summarize these reasonings in the following proposition:

Proposition 5. *Let*

$$\begin{aligned} \Lambda & := \frac{1}{2} \text{ad } Y_3(e_1) & e_{2\lambda,\alpha} & := \frac{1}{2} \text{ad } Y_3(e_\alpha) & f_{2\lambda,\alpha} & := \varkappa A_\alpha \\ e_{\lambda,i} & := -\frac{1}{2} \text{ad } Y_2(\bar{e}_i) & f_{\lambda,i} & := \frac{1}{2} \text{ad } Y_1(e_i) & \tilde{A}_{\alpha\beta} & := \varkappa A_{\alpha\beta} \end{aligned}$$

where Latin indices vary from 0 to 7 and Greek ones (except λ) vary from 1 to 7. We have the following commutator relations:

$$\begin{aligned} [\Lambda, e_{2\lambda,\alpha}] &= -f_{2\lambda,\alpha} & [\Lambda, f_{2\lambda,\alpha}] &= e_{2\lambda,\alpha} & [\Lambda, e_{\lambda,i}] &= -\frac{1}{2}f_{\lambda,i} & [\Lambda, f_{\lambda,i}] &= \frac{1}{2}e_{\lambda,i} \\ [\Lambda, \tilde{A}_{\alpha\beta}] &= 0 & [e_{2\lambda,\alpha}, e_{2\lambda,\beta}] &= \tilde{A}_{\beta\alpha} & [e_{2\lambda,\alpha}, f_{2\lambda,\beta}] &= -\delta_{\alpha\beta}\Lambda & [f_{2\lambda,\alpha}, f_{2\lambda,\beta}] &= \tilde{A}_{\beta\alpha} \\ [e_{2\lambda,\alpha}, e_{\lambda,j}] &= \frac{1}{2}f_{\lambda,e_\alpha e_j} & [e_{2\lambda,\alpha}, f_{\lambda,j}] &= \frac{1}{2}e_{\lambda,e_\alpha e_j} & [f_{2\lambda,\alpha}, e_{\lambda,j}] &= -\frac{1}{2}e_{\lambda,e_\alpha e_j} \\ [f_{2\lambda,\alpha}, f_{\lambda,j}] &= \frac{1}{2}f_{\lambda,e_\alpha e_j} & [e_{\lambda,i}, e_{\lambda,j}] &= \frac{1}{4}\varkappa C_{2,\bar{e}_i,\bar{e}_j} = \frac{1}{2}f_{2\lambda,e_i\bar{e}_j} + \frac{1}{2}\varkappa\tilde{C}_{2,i,j} & & i \neq j \\ [f_{\lambda,i}, f_{\lambda,j}] &= \frac{1}{4}\varkappa C_{1,e_i,e_j} = -\frac{1}{2}f_{2\lambda,e_i\bar{e}_j} + \frac{1}{2}\varkappa\tilde{C}_{1,i,j} & & i \neq j \\ [e_{\lambda,i}, f_{\lambda,j}] &= \begin{cases} -\frac{1}{2}\Lambda & i = j \\ -\frac{1}{2}e_{2\lambda,e_i\bar{e}_j} & i \neq j \end{cases} \end{aligned}$$

where we denote $f_{\lambda,e_\alpha e_j} := f_{\lambda,i}$ if $e_\alpha e_j = e_i$ and $f_{\lambda,e_\alpha e_j} := -f_{\lambda,i}$ if $e_\alpha e_j = -e_i$. We use analogous notation for $e_{\lambda,i}, e_{2\lambda,\gamma}, f_{2\lambda,\gamma}$. Here operators $\tilde{C}_{l,i,j}, l = 1, 2, i \neq j$ are in \mathfrak{k}_0 and act as

$$\begin{aligned} \tilde{C}_{1,i,j}(e_k) &= e_k e_i \cdot \bar{e}_j & e_k \neq 1, \pm e_i \bar{e}_j & \quad \tilde{C}_{1,i,j}(e_k) = 0 & e_k = 1, \pm e_i \bar{e}_j \\ \tilde{C}_{2,i,j}(e_k) &= e_j \cdot \bar{e}_i e_k & e_k \neq 1, \pm e_i \bar{e}_j & \quad \tilde{C}_{2,i,j}(e_k) = 0 & e_k = 1, \pm e_i \bar{e}_j. \end{aligned}$$

The chosen bases $\Lambda, e_{\lambda,i}, e_{2\lambda,\alpha}, f_{\lambda,i}, f_{2\lambda,\alpha}$ in spaces $\mathfrak{a}, \mathfrak{p}_\lambda, \mathfrak{p}_{2\lambda}, \mathfrak{k}_\lambda, \mathfrak{k}_{2\lambda}$ correspond to proposition 1.

Proof. The commutator relations are easy consequences of (57), (58), lemma 1 and relations in the algebra $\mathfrak{k} \simeq \mathfrak{so}(8)$. For example, let us calculate the commutator $[f_{2\lambda,\alpha}, e_{\lambda,j}]$. Actually,

from (57) and lemma 1 we obtain

$$\begin{aligned} [f_{2\lambda,\alpha}, e_{\lambda,j}] &= -[\kappa A_\alpha, \frac{1}{2} \text{ad } Y_2(\bar{e}_j)] = -\frac{1}{2} \text{ad } Y_2(A_\alpha^{(2)} \bar{e}_j) = -\frac{1}{2} \text{ad } Y_2(R_\alpha \bar{e}_j) \\ &= -\frac{1}{4} \text{ad } Y_2(\bar{e}_j e_\alpha) = \frac{1}{4} \text{ad } Y_2(\bar{e}_j \bar{e}_\alpha) = -\frac{1}{2} e_{\lambda, e_\alpha e_j}. \end{aligned}$$

Similar calculations are also valid for $[f_{2\lambda,\alpha}, f_{\lambda,j}]$.

Now, let us calculate $[e_{\lambda,i}, e_{\lambda,j}]$, $i \neq j$. From (58) we obtain

$$[e_{\lambda,i}, e_{\lambda,j}] = \frac{1}{4} [\text{ad } Y_2(\bar{e}_i), \text{ad } Y_2(\bar{e}_j)] = \frac{1}{4} \kappa C_{2,\bar{e}_i,\bar{e}_j} \quad i \neq j.$$

From (47) and (59) we obtain

$$\frac{1}{2} C_{2,\bar{e}_i,\bar{e}_j}(e_k) = \frac{1}{2} (e_j \cdot \bar{e}_i e_k - e_i \cdot \bar{e}_j e_k) = -e_i \cdot \bar{e}_j e_k.$$

In particular,

$$\frac{1}{2} C_{2,\bar{e}_i,\bar{e}_j}(1) = -e_i \bar{e}_j \quad \frac{1}{2} C_{2,\bar{e}_i,\bar{e}_j}(e_i \bar{e}_j) = -(e_i \cdot \bar{e}_j)^2 = 1$$

so

$$\frac{1}{2} \kappa C_{2,\bar{e}_i,\bar{e}_j} = \kappa A_{e_i \bar{e}_j} + \kappa \tilde{C}_{2,i,j} = f_{2\lambda, e_i \bar{e}_j} + \kappa \tilde{C}_{2,i,j}$$

where $\tilde{C}_{2,i,j} \in \mathfrak{k}_0$ and

$$\tilde{C}_{2,i,j}(e_k) = e_j \cdot \bar{e}_i e_k \quad e_k \neq 1, \pm e_i \bar{e}_j \quad \tilde{C}_{2,i,j}(e_i \bar{e}_j) = \tilde{C}_{2,i,j}(1) = 0.$$

Similar calculations are also valid for $[f_{\lambda,i}, f_{\lambda,j}]$. \square

11.2. Invariants in $S(\mathfrak{a} \oplus \mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda})$

Invariant operators D_0, \dots, D_6 , corresponding to some K_0 -invariants in $S(\mathfrak{a} \oplus \mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda})$ are already constructed in section 4. Here we shall construct other independent invariants of K_0 -action on $S(\mathfrak{a} \oplus \mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda})$ or equivalently from $S(\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda})$, since \mathfrak{a} is an invariant one-dimensional space, and corresponding invariant differential operators.

An element $\Phi \in K'$ is from $K_0 \subset K'$ iff $\Phi_3(1) = 1$ and then $\Phi_3(\xi) = \xi$ for any $\xi \in \mathbb{R} \subset \mathbb{C}a_3$. Hereafter in this section $\Phi \in K_0$. The orthogonality of Φ_i means that

$$\text{Re}(\Phi_i(\xi) \overline{\Phi_i(\eta)}) = \text{Re}(\xi \bar{\eta}) \quad \xi, \eta \in \mathbb{C}a_i. \quad (61)$$

In particular, $\Phi_i(\xi) \overline{\Phi_i(\xi)} = |\xi|^2$ and

$$\overline{\Phi_i(\xi)^{-1}} = \Phi_i(\xi) / |\xi|^2. \quad (62)$$

For $\eta = \bar{\xi}$ from (55) we obtain $\Phi_1(\xi) \Phi_2(\bar{\xi}) = \overline{\Phi_3(|\xi|^2)} = |\xi|^2$, so from (62) $\Phi_1(\xi) = |\xi|^2 \Phi_2(\bar{\xi})^{-1} = \overline{\Phi_2(\bar{\xi})}$ and

$$\Phi_1 = \iota \circ \Phi_2 \circ \iota. \quad (63)$$

Let $Q_1(\xi, \eta) = \text{Re}(\xi \bar{\eta})$, $\xi \in \mathbb{C}a_1$, $\eta \in \mathbb{C}a_2$. From (61) and (63) we get

$$Q_1(\Phi_1(\xi), \Phi_2(\eta)) = \text{Re}(\Phi_1(\xi) \Phi_2(\eta)) = \text{Re}(\Phi_1(\xi) \overline{\Phi_1(\bar{\eta})}) = \text{Re}(\xi \bar{\eta}) = Q_1(\xi, \eta).$$

Thus, $Q_1(\xi, \eta)$ is invariant under the K_0 -action.

From proposition 3, it follows that $\Phi_1 = g^L$, $\Phi_2 = g^R$, $\Phi_3 = \iota \circ g^V \circ \iota = g^V$, where g^L, g^R, g^V are respectively left spinor, right spinor and vector representation of the group $K_0 \simeq \text{Spin}(7)$, since $\iota|_{\mathbb{C}a'_3} = -\text{id}$. Besides, the K_0 -action on $\text{Im}(\xi \eta)$, $\xi \in \mathbb{C}a_1, \eta \in \mathbb{C}a_2$ equals g^V , so $Q_2(\xi, \eta, \zeta) := \text{Re}(\text{Im}(\xi \eta) \zeta)$ is invariant under K_0 -action for $\zeta \in \mathbb{C}a'_3$.

According to section 4 the K_0 -action on \mathfrak{p}_λ is equivalent to the K_0 -action on \mathfrak{k}_λ and the K_0 -action on $\mathfrak{p}_{2\lambda}$ is equivalent to the K_0 -action on $\mathfrak{k}_{2\lambda}$. This equivalence is established by the correspondence of bases $e_{\lambda,i} \leftrightarrow f_{\lambda,i}$ and $e_{2\lambda,\alpha} \leftrightarrow f_{2\lambda,\alpha}$. It is also confirmed by the

formulae $[Y_2(\xi), E_1] = X_2(\xi)$, $\text{ad } Y_1(\xi)(X_3(1)) = -X_2(\bar{\xi})$, $\xi \in \mathbb{C}a$ and (63). Therefore the analogue of $\text{Im}(\xi\eta)$, $\xi \in \mathbb{C}a_1$, $\eta \in \mathbb{C}a_2$ in $S(\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda}) \otimes \mathbb{C}a$ is

$$\sum_{i \neq j} f_{\lambda,i} e_{\lambda,\bar{e}_j} \otimes e_i e_j.$$

Thus, after the identification $\mathfrak{p}_{2\lambda} \simeq \mathbb{C}a'_3$ the invariant Q_2 gives the invariant from $S(\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda})$

$$\sum_{i \neq j} f_{\lambda,i} e_{\lambda,\bar{e}_j} e_{2\lambda,e_i} e_j = \sum_{i \neq j} f_{\lambda,i} e_{\lambda,j} e_{2\lambda,e_i \bar{e}_j}.$$

Therefore, we can define the invariant differential operator:

$$D_7 = -\frac{1}{4} \sum_{i \neq j} \{ \{ f_{\lambda,i}, e_{\lambda,j} \}, e_{2\lambda,e_i \bar{e}_j} \} = \frac{1}{4} \sum_{i \neq j} \{ \{ f_{\lambda,j}, e_{\lambda,i} \}, e_{2\lambda,e_i \bar{e}_j} \}.$$

Similarly, the identification $\mathfrak{k}_{2\lambda} \simeq \mathbb{C}a'_3$ gives the invariant differential operator:

$$D_8 = -\frac{1}{4} \sum_{i \neq j} \{ \{ f_{\lambda,i}, e_{\lambda,j} \}, f_{2\lambda,e_i \bar{e}_j} \} = \frac{1}{4} \sum_{i \neq j} \{ \{ f_{\lambda,j}, e_{\lambda,i} \}, f_{2\lambda,e_i \bar{e}_j} \}.$$

It is clear that equation (55) remains valid after the cyclic permutation of indices 1, 2, 3:

$$\begin{aligned} \Phi_3(\zeta)\Phi_1(\xi) &= \overline{\Phi_2(\zeta\bar{\xi})} & \Phi_2(\eta)\Phi_3(\zeta) &= \overline{\Phi_1(\eta\bar{\zeta})} \\ \xi \in \mathbb{C}a_1 & & \eta \in \mathbb{C}a_2 & & \zeta \in \mathbb{C}a'_3. \end{aligned} \quad (64)$$

Define

$$P(\xi, \eta, \zeta_1, \zeta_2) := \text{Re}(\zeta_1 \xi \cdot \eta \zeta_2) \quad \zeta_1, \zeta_2 \in \mathbb{C}a'_3.$$

The function $P(\xi, \eta, \zeta_1, \zeta_2)$ is invariant w.r.t. the K_0 -action, since due to (64), (63) and (61):

$$\begin{aligned} P(\Phi_1(\xi), \Phi_2(\eta), \Phi_3(\zeta_1), \Phi_3(\zeta_2)) &= \text{Re}(\Phi_3(\zeta_1)\Phi_1(\xi) \cdot \Phi_2(\eta)\Phi_3(\zeta_2)) \\ &= \text{Re}(\overline{\Phi_2(\zeta_1\bar{\xi})}\Phi_1(\eta\bar{\zeta}_2)) = \text{Re}(\Phi_1(\zeta_1\xi)\overline{\Phi_1(\eta\bar{\zeta}_2)}) = \text{Re}(\zeta_1\xi \cdot \overline{\eta\bar{\zeta}_2}) \\ &= P(\xi, \eta, \zeta_1, \zeta_2). \end{aligned}$$

Functions $P(\xi, \eta, \zeta_1, \zeta_2)$ and $P(\xi, \eta, \zeta_2, \zeta_1)$ are not independent. Indeed, the corollary 15.12 in [18] gives

$$\begin{aligned} \text{Re}(ab \cdot c) &= \text{Re}(bc \cdot a) = \text{Re}(ca \cdot b) = \text{Re}(a \cdot bc) = \text{Re}(b \cdot ca) = \text{Re}(c \cdot ab) \\ a, b, c &\in \mathbb{C}a. \end{aligned}$$

Therefore, using the Moufang identity (46) we obtain

$$\begin{aligned} P(\xi, \eta, \zeta, \zeta) &= \text{Re}(\zeta \cdot \xi\eta \cdot \zeta) = \text{Re}(\zeta^2 \cdot \xi\eta) = -\text{Re}(|\zeta|^2 \xi\eta) = -|\zeta|^2 \text{Re}(\xi\eta) \\ &= -|\zeta|^2 Q_1(\xi, \eta) \end{aligned} \quad (65)$$

which means that for $\zeta_1 = \zeta_2 = \zeta$ invariant $P(\xi, \eta, \zeta, \zeta)$ is expressed through invariants of the second order. Using the polarization of (65) w.r.t. ζ , which means the substitution $\zeta = \zeta_1 + \zeta_2$, we get

$$P(\xi, \eta, \zeta_1, \zeta_2) + P(\xi, \eta, \zeta_2, \zeta_1) = -2\langle \zeta_1, \zeta_2 \rangle Q_1(\xi, \eta).$$

This means the dependence of two invariants $P(\xi, \eta, \zeta_1, \zeta_2)$, $P(\xi, \eta, \zeta_2, \zeta_1)$ and invariants $Q_1(\xi, \eta)$, $\langle \zeta_1, \zeta_2 \rangle$ is of the second order. The last two invariants correspond to operators D_3 and D_6 .

For constructing the invariant differential operator D_9 we shall use the invariant function

$$P(\xi, \eta, \zeta_1, \zeta_2) - P(\xi, \eta, \zeta_2, \zeta_1).$$

Using $\sum_k e_{\lambda, \bar{e}_k} \otimes e_k$ as the analogue of η we get the corresponding expression from $S(\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda})$:

$$\sum_{\substack{i \neq j \\ j \neq k}} (f_{2\lambda, e_j \bar{e}_i} f_{\lambda, i} e_{\lambda, \bar{e}_k} e_{2\lambda, \bar{e}_k \bar{e}_j} - e_{2\lambda, e_j \bar{e}_i} f_{\lambda, i} e_{\lambda, \bar{e}_k} f_{2\lambda, \bar{e}_k \bar{e}_j})$$

$$= \sum_{\substack{i \neq j \\ j \neq k}} (e_{2\lambda, e_i \bar{e}_j} f_{\lambda, i} e_{\lambda, k} f_{2\lambda, e_k \bar{e}_j} - f_{2\lambda, e_i \bar{e}_j} f_{\lambda, i} e_{\lambda, k} e_{2\lambda, e_k \bar{e}_j})$$

since for $i \neq j$ it holds that $e_j \bar{e}_i = -\overline{e_j \bar{e}_i} = -e_i \bar{e}_j$.

Define the corresponding invariant differential operator as

$$D_9 = \frac{1}{8} \sum_{\substack{i \neq j \\ j \neq k}} (\{ \{ e_{2\lambda, e_i \bar{e}_j}, f_{\lambda, i} \}, \{ f_{2\lambda, e_k \bar{e}_j}, e_{\lambda, k} \} \} - \{ \{ e_{2\lambda, e_i \bar{e}_j}, e_{\lambda, i} \}, \{ f_{2\lambda, e_k \bar{e}_j}, f_{\lambda, k} \} \} \}).$$

Let us show that there are exactly nine independent K_0 -invariants in $S(\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda})$.

Indeed, $\dim(\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda}) = 8 + 8 + 7 + 7 = 30$ and $\dim K_0 = \dim \text{Spin}(7) = 21$. Therefore, the codimension of K_0 -orbits in $\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda}$ is at least $30 - 21 = 9$ and there should be at least nine independent K_0 -invariants in $S(\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda})$.

On the other hand, it is obvious that the stationary subgroup, corresponding to a point in a general position, of the group $\text{Spin}(7)$, acting on $\mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda}$ by $g^V \oplus g^V$, is $\text{Spin}(5)$. Therefore, the dimension of general $\text{Spin}(7)$ -orbits in $\mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda}$ is $\dim \text{Spin}(7) - \dim \text{Spin}(5) = 11$. The group $\text{Spin}(5)$ is isomorphic to $U_{\mathbb{H}}(2)$, see [18], proposition 5.1. In section 6 the six independent invariants of the diagonal $U_{\mathbb{H}}(2)$ -action on $\mathbb{H}^2 \oplus \mathbb{H}^2 \simeq \mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda$ were found, so general orbits of the last action are ten dimensional, since $\dim_{\mathbb{R}}(\mathbb{H}^2 \oplus \mathbb{H}^2) - 6 = 10$. Thus, general $\text{Spin}(7)$ -orbits in $\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda}$ are $11 + 10 = 21$ dimensional, their codimension is 9 and there are exactly nine functionally independent invariants of $\text{Spin}(7)$ -action on $\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda}$.

It is not known if there are any other invariants of this action, which are polynomial in $e_{\lambda, i}, f_{\lambda, i}, e_{2\lambda, \alpha}, f_{2\lambda, \alpha}$ and are not polynomial in D_1, \dots, D_9 . Such invariants should be connected with D_1, \dots, D_9 by algebraic equation of a degree greater than 1. In the case of $\mathbf{P}^n(\mathbb{H})_{\mathbb{S}}, n \geq 3$ there is the invariant D_{10} which does not admit the polynomial expression in D_1, \dots, D_9 , but D_{10}^2 does admit such an expression. The operator D_{10} arises in commutative relations of D_1, \dots, D_9 .

In the next section it is found that all commutators of operators D_1, \dots, D_9 in the octonionic case are polynomial in D_1, \dots, D_9 . Therefore it seems probable that there is no analogue of D_{10} in the octonionic case.

It is easily verified that automorphisms ζ_α, σ acts on D_7, D_8, D_9 as

$$\begin{aligned} \zeta_\alpha(D_7) &= \cos(\alpha)D_7 - \sin(\alpha)D_8 & \zeta_\alpha(D_8) &= \sin(\alpha)D_7 + \cos(\alpha)D_8 \\ \zeta_\alpha(D_9) &= D_9 & \sigma(D_7) &= D_7 & \sigma(D_8) &= -D_8 & \sigma(D_9) &= D_9. \end{aligned}$$

Similar to the previous sections, in order to get the generators of the algebra $\text{Diff}(\mathbb{H}^2(\mathbb{C}a)_{\mathbb{S}})$ one can use remark 1 and make the formal substitution:

$$\Lambda \rightarrow \mathbf{i}\Lambda \quad e_{\lambda, i} \rightarrow \mathbf{i}e_{\lambda, i} \quad f_{\lambda, i} \rightarrow f_{\lambda, i} \quad e_{2\lambda, \alpha} \rightarrow \mathbf{i}e_{\lambda, \alpha} \quad f_{\lambda, \alpha} \rightarrow f_{\lambda, \alpha}.$$

This substitution produces the following substitution for the generators D_0, \dots, D_{10} :

$$\begin{aligned} D_0 &\rightarrow \mathbf{i}\bar{D}_0 & D_1 &\rightarrow -\bar{D}_1 & D_2 &\rightarrow \bar{D}_2 & D_3 &\rightarrow \mathbf{i}\bar{D}_3 & D_4 &\rightarrow -\bar{D}_4 \\ D_5 &\rightarrow \bar{D}_5 & D_6 &\rightarrow \mathbf{i}\bar{D}_6 & D_7 &\rightarrow -\bar{D}_7 & D_8 &\rightarrow \mathbf{i}\bar{D}_8 & D_9 &\rightarrow -\bar{D}_9. \end{aligned} \tag{66}$$

The operators $\bar{D}_0, \dots, \bar{D}_9$ generate the algebra $\text{Diff}(\mathbb{H}^2(\mathbb{C}a)_{\mathbb{S}})$.

12. Relations in algebras $\text{Diff}(\mathbf{P}^2(\mathbb{C}a)_{\mathbb{S}})$ and $\text{Diff}(\mathbf{H}^2(\mathbb{C}a)_{\mathbb{S}})$

Below there are all 45 commutative relations of operators D_0, \dots, D_9 . An example of a calculation of such a relation is given in appendix A. All methods described in section 7 for calculating commutative relations were used in this case. Besides, the numeration of the base elements $e_{\lambda,i}, f_{\lambda,i}, e_{2\lambda,\alpha}, f_{\lambda,\alpha}$ by octonionic units $e_i, i = 0, \dots, 7$ is very convenient

$$\begin{aligned}
[D_0, D_1] &= -D_3 & [D_0, D_2] &= D_3 & [D_0, D_3] &= \frac{1}{2}(D_1 - D_2) & [D_0, D_4] &= -2D_6 \\
[D_0, D_5] &= 2D_6 & [D_0, D_6] &= D_4 - D_5 & [D_0, D_7] &= -D_8 & [D_0, D_8] &= D_7 \\
[D_0, D_9] &= 0 & [D_1, D_2] &= -\{D_0, D_3\} - 2D_7 & [D_1, D_3] &= -\frac{1}{2}\{D_0, D_1\} + D_8 + 10D_0 \\
[D_1, D_4] &= 2D_7 & [D_1, D_5] &= 0 & [D_1, D_6] &= D_8 \\
[D_1, D_7] &= \frac{1}{2}\{D_1, D_2 - D_4\} - D_9 - \frac{1}{2}\{D_3, D_6\} - D_3^2 - 5D_0^2 - \frac{3}{32}D_1 - \frac{283}{32}D_2 + \frac{19}{2}D_4 - \frac{1}{2}D_5 \\
[D_1, D_8] &= -\frac{1}{2}\{D_3, D_5\} - \frac{1}{2}\{D_1, D_6\} + 10D_6 + \frac{35}{4}D_3 \\
[D_1, D_9] &= \frac{1}{2}\{D_5, D_7\} - \frac{1}{2}\{D_6, D_8\} - \frac{189}{32}\{D_0, D_3\} - \frac{169}{16}D_7 \\
[D_2, D_3] &= \frac{1}{2}\{D_0, D_2\} + D_8 - 10D_0 \\
[D_2, D_4] &= -2D_7 & [D_2, D_5] &= 0 & [D_2, D_6] &= -D_8 \\
[D_2, D_7] &= -\frac{1}{2}\{D_2, D_1 - D_4\} + D_9 - \frac{1}{2}\{D_3, D_6\} + D_3^2 + 5D_0^2 + \frac{3}{32}D_2 + \frac{283}{32}D_1 - \frac{19}{2}D_4 + \frac{1}{2}D_5 \\
[D_2, D_8] &= \frac{1}{2}\{D_2, D_6\} - \frac{1}{2}\{D_3, D_5\} + \frac{35}{4}D_3 - 10D_6 \\
[D_2, D_9] &= -\frac{1}{2}\{D_5, D_7\} + \frac{1}{2}\{D_6, D_8\} + \frac{189}{32}\{D_0, D_3\} + \frac{169}{16}D_7, [D_3, D_4] = 0 \\
[D_3, D_5] &= 2D_8 & [D_3, D_6] &= D_7 & [D_3, D_7] &= -\frac{1}{4}\{D_1 + D_2, D_6\} + 10D_6 \\
[D_3, D_8] &= \frac{1}{2}\{D_1, D_2\} - \frac{1}{4}\{D_1 + D_2, D_5\} - D_9 - D_3^2 - 5D_0^2 - \frac{143}{32}(D_1 + D_2) - \frac{1}{2}D_4 + \frac{19}{2}D_5 \\
[D_3, D_9] &= \frac{1}{2}\{D_4, D_8\} - \frac{1}{2}\{D_6, D_7\} + \frac{189}{64}\{D_0, D_1 - D_2\} - \frac{169}{16}D_8 \\
[D_4, D_5] &= -2\{D_0, D_6\} & [D_4, D_6] &= -\{D_0, D_4\} + \frac{35}{2}D_0 \\
[D_4, D_7] &= \frac{1}{2}\{D_1 - D_2, D_4\} + \frac{35}{4}(D_2 - D_1) & [D_4, D_8] &= \frac{1}{2}\{D_1 - D_2, D_6\} - \{D_0, D_7\} \\
[D_4, D_9] &= -9\{D_0, D_6\} & [D_5, D_6] &= \{D_0, D_5\} - \frac{35}{2}D_0 \\
[D_5, D_7] &= \{D_3, D_6\} + \{D_0, D_8\} & [D_5, D_8] &= \{D_3, D_5\} - \frac{35}{2}D_3 & [D_5, D_9] &= 9\{D_0, D_6\} \\
[D_6, D_7] &= \frac{1}{4}\{D_1 - D_2, D_6\} + \frac{1}{2}\{D_3, D_4\} + \frac{1}{2}\{D_0, D_7\} - \frac{35}{4}D_3 \\
[D_6, D_8] &= \frac{1}{4}\{D_1 - D_2, D_5\} + \frac{1}{2}\{D_3, D_6\} - \frac{1}{2}\{D_0, D_8\} + \frac{35}{8}(D_2 - D_1) \\
[D_6, D_9] &= \frac{9}{2}\{D_0, D_4 - D_5\} \\
[D_7, D_8] &= -\frac{1}{4}\{D_0, \{D_1, D_2\}\} + \frac{1}{2}\{D_0, D_3^2\} + \frac{1}{2}\{D_0, D_9\} + \frac{1}{4}\{D_1 - D_2, D_8\} + \frac{1}{4}\{D_0, D_5\} \\
&\quad + \frac{283}{64}\{D_0, D_1 + D_2\} - \frac{175}{2}D_0 - \frac{1}{2}\{D_3, D_7\} + 5D_0^3 + \frac{1}{4}\{D_0, D_4\} \\
[D_7, D_9] &= \frac{1}{4}\{\{D_0, D_7\}, D_6\} + \frac{1}{8}\{D_2 - D_1, \{D_4, D_5\}\} \\
&\quad - \frac{1}{4}\{\{D_0, D_4\}, D_8\} + \frac{1}{4}\{D_1 - D_2, D_6^2\} - \frac{1}{2}\{D_0, D_8\} + \frac{25}{32}\{D_3, D_6\} \\
&\quad + \frac{185}{64}\{D_1 - D_2, D_4\} + \frac{17}{8}\{D_1 - D_2, D_5\} + \frac{35 \times 181}{128}(D_2 - D_1)
\end{aligned}$$

$$\begin{aligned}
[D_8, D_9] = & -\frac{1}{4}\{\{D_0, D_6\}, D_8\} - \frac{1}{4}\{D_3, \{D_4, D_5\}\} + \frac{1}{4}\{\{D_0, D_7\}, D_5\} + \frac{1}{2}\{D_3, D_6^2\} \\
& + \frac{169}{32}\{D_3, D_5\} + \frac{45}{64}\{D_1 - D_2, D_6\} + \frac{37}{8}\{D_3, D_4\} + \frac{5}{8}\{D_0, D_7\} - \frac{35 \times 177}{64}D_3.
\end{aligned}$$

Using these relations it is not difficult to verify that the operator $D^* = D_0^2 + D_1 + D_2 + D_4 + D_5$ lies in the centre of the algebra $\text{Diff}(\mathbf{P}^n(\mathbb{H})_{\mathbb{S}})$ in accordance with section 4.

Using substitution (66) one can obtain from the above relations the commutative relations for the algebra $\text{Diff}(\mathbf{H}^2(\mathbb{C}a)_{\mathbb{S}})$.

13. Connection of algebras $\text{Diff}(M_{\mathbb{S}})$ with the two-body problem

In [5] an expression of the quantum two-body Hamiltonian with a central potential $V(\rho)$ on an arbitrary two-point homogeneous space M through radial differential operators and generators of an isometry group was found. Using the notation of the present paper we can write these expressions in the following way:

$$\hat{H} = L_2 + \{L_1, D_0\} + a_0 D_0^2 + \sum_{i=1}^6 a_i D_i + V(\rho)$$

for $M = \mathbf{P}^n(\mathbb{H})$ and $M = \mathbf{P}^2(\mathbb{C}a)$;

$$\hat{H} = L_2 + \{L_1, D_0\} + a_0 D_0^2 + \sum_{i=1}^3 a_i D_i + a_4 D_4^2 + a_5 D_5^2 + a_6 \{D_4, D_5\} + V(\rho)$$

for $M = \mathbf{P}^n(\mathbb{C})$ and

$$\hat{H} = L_2 + \{L_1, D_0\} + a_0 D_0^2 + \sum_{i=1}^3 a_i D_i + V(\rho)$$

for $M = \mathbf{P}^n(\mathbb{R}), \mathbf{S}^n, n \geq 3$ and

$$\hat{H} = L_2 + \{L_1, D_0\} + a_0 D_0^2 + a_1 D_1^2 + a_2 D_2^2 + \frac{1}{2} a_3 \{D_1, D_2\} + V(\rho),$$

for $M = \mathbf{P}^2(\mathbb{R}), \mathbf{S}^2$.

Here ρ is the distance between particles, $L_i, i = 1, 2$ is some ordinary differential operator of the i th order w.r.t. ρ , $a_0 = \text{const}$, $a_i, i = 1, \dots, 6$ are some functions of ρ and masses of particles. The analogous expressions for noncompact spaces can be obtained by substitutions $D_i \rightarrow \bar{D}_i$ from above.

The main difference of these expressions from the Euclidean case is the presence of noncommutative operators with coefficients depending on ρ . This difference makes the two-body problem on M quite difficult. However, every common eigenfunction of generators D_i gives an isolated ordinary differential equation for a radial part of an eigenfunction for \hat{H} . Using this approach some exact spectral series for the two-body problem on \mathbf{S}^n were found for several potentials in [15]. For other two-point compact homogeneous spaces similar calculations should be more difficult.

Appendix A. Calculation of some commutative relations

In this appendix we shall illustrate the main ideas of calculating some commutative relations. Let us start from commutative relations (24) from section 7. We shall obtain some relations requiring the minimal calculations.

Let operators D_0, \dots, D_{10} be defined as in section 6. First let us consider the commutator $[D_1, D_4]$. It is not difficult to verify the following equalities for elements A, B, C of an arbitrary associative algebra:

$$[A, \{B, C\}] = \{[A, B], C\} + \{B, [A, C]\} \quad (\text{A.1})$$

$$\{\{A, B\}, C\} - \{A, \{B, C\}\} = [B, [A, C]] \quad (\text{A.2})$$

$$\{\{A, B\}, C\} = 2\{B, C\}A + \{[A, B], C\} + \{[A, C], B\} + [B, [A, C]]. \quad (\text{A.3})$$

In particular, when $C = B$, from (A.1) we have $[A, B^2] = \{[A, B], B\}$. This implies

$$[D_1, D_4] = \{[D_1, \Upsilon_{12}], \Upsilon_{12}\} + \{[D_1, \Omega_{12}], \Omega_{12}\} + \{[D_1, \Theta_{12}], \Theta_{12}\}.$$

Using (12) and (A.1) again, we obtain

$$[D_1, \Upsilon_{12}] = \square_1 \quad [D_1, \Omega_{12}] = \square_2 \quad [D_1, \Theta_{12}] = \square_3.$$

Thus

$$[D_1, D_4] = \{\square_1, \Upsilon_{12}\} + \{\square_2, \Omega_{12}\} + \{\square_3, \Theta_{12}\} = 2D_7. \quad (\text{A.4})$$

Using the permutation of coordinates z_1 and z_2 (or equivalently the automorphism $\sigma \circ \zeta_\pi$, see section 6), we obtain from (A.4):

$$[D_2, D_4] = -2D_7.$$

Suppose now we already know the expressions for commutators

$$[D_0, D_1], [D_0, D_3], [D_0, D_7], [D_1, D_2], [D_1, D_4], [D_1, D_5], [D_1, D_6], [D_1, D_7] \\ [D_1, D_8], [D_2, D_6], [D_3, D_4], [D_3, D_6], [D_4, D_5], [D_4, D_6], [D_4, D_8].$$

Then from the Jacobi identity and (A.1) we have

$$0 = [D_1, [D_8, D_4]] + [D_4, [D_1, D_8]] + [D_8, [D_4, D_1]] = [D_1, \frac{1}{2}\{D_2 - D_1, D_6\} + \{D_0, D_7\}] \\ + [D_4, n(n-1)D_6 - \frac{1}{2}\{D_3, D_5\} + \frac{3}{4}D_3 - \frac{1}{2}\{D_1, D_6\}] - 2[D_8, D_7] \\ = \frac{1}{2}\{[D_1, D_2], D_6\} - \frac{1}{2}\{D_1 - D_2, [D_1, D_6]\} + \{[D_1, D_0], D_7\} + \{D_0, [D_1, D_7]\} \\ + n(n-1)[D_4, D_6] - \frac{1}{2}\{[D_4, D_3], D_5\} - \frac{1}{2}\{D_3, [D_4, D_5]\} + \frac{3}{4}[D_4, D_3] \\ - \frac{1}{2}\{D_1, [D_4, D_6]\} - \frac{1}{2}\{[D_4, D_1], D_6\} - 2[D_8, D_7] = -\frac{1}{2}\{[D_3, D_0], D_6\} \\ - [D_4, D_6] + \frac{1}{2}\{D_2 - D_1, D_8\} + \{D_3, D_7\} + \{D_0, n(n-1)D_4 - \frac{1}{2}\{D_3, D_6\}\} \\ - \frac{1}{2}\{D_1, D_4\} + \frac{3}{8}(D_1 - D_2) + D_9 + D_{10} - n(n-1)\{D_0, D_4\} + \frac{3}{2}n(n-1)D_0 \\ + \{D_3, \{D_6, D_0\}\} + \frac{1}{2}\{D_1, \{D_0, D_4\}\} - \frac{3}{4}\{D_1, D_0\} + \{D_7, D_6\} - 2[D_8, D_7] \\ = \frac{1}{2}\{D_2 - D_1, D_8\} + \{D_3, D_7\} + \frac{3}{8}\{D_0, D_1 - D_2\} + \{D_0, D_9 + D_{10}\} \\ + \frac{3}{2}n(n-1)D_0 - \frac{3}{4}\{D_0, D_1\} - 2[D_8, D_7].$$

In the last equality we took into account the formulae

$$\{\{D_6, D_0\}, D_3\} - \{D_6, \{D_0, D_3\}\} + \{\{D_0, D_6\}, D_3\} - \{D_0, \{D_6, D_3\}\} = [D_0, [D_6, D_3]] \\ + [D_6, [D_0, D_3]] = -[D_0, D_7] - \frac{1}{2}[D_6, D_2 - D_1] = D_8 + \frac{1}{2}(-D_8 - D_8) = 0 \\ \{\{D_0, D_4\}, D_1\} - \{D_0, \{D_4, D_1\}\} = [D_4, [D_0, D_1]] = [D_4, D_3] = 0$$

which are consequences of (A.2).

Thus we get

$$[D_7, D_8] = \frac{1}{4}\{D_1 - D_2, D_8\} - \frac{1}{2}\{D_3, D_7\} + \frac{3}{16}\{D_0, D_1 + D_2\} - \frac{1}{2}\{D_0, D_9 + D_{10}\} \\ - \frac{3}{4}n(n-1)D_0.$$

Now let us demonstrate the calculation modulo $(U(\mathfrak{g})\mathfrak{k}_0)^{K_0}$. Let D_0, \dots, D_3 are generators of $\text{Diff}(\mathbf{S}_{\mathbb{S}}^n)$, $n \geq 3$, $\mathfrak{g} = \mathfrak{so}(n+1)$, $\mathfrak{k}_0 = \mathfrak{so}(n-1)$, $K_0 = SO(n-1)$. Then from (A.1) we obtain

$$\begin{aligned} [D_1, D_3] &= -8 \sum_{k,l=3}^{n+1} (\{\{\Psi_{1k}, [\Psi_{1k}, \Psi_{1l}], \Psi_{2l}\} + \{\Psi_{1l}, \{\Psi_{1k}, [\Psi_{1k}, \Psi_{2l}]\}\}) \\ &= 4 \sum_{k,l=3}^{n+1} (\{\{\Psi_{1k}, \Psi_{kl}\}, \Psi_{2l}\} + \delta_{kl} \{\Psi_{1l}, \{\Psi_{1k}, \Psi_{12}\}\}) \\ &= 4 \sum_{\substack{k,l=3 \\ k \neq l}}^{n+1} \{\{\Psi_{kl}, \Psi_{1k}\}, \Psi_{2l}\} + 4 \sum_{k=3}^{n+1} \{\Psi_{1k}, \{\Psi_{1k}, \Psi_{12}\}\}. \end{aligned} \quad (\text{A.5})$$

From formula (A.3) and commutative relations (12) one obtains

$$\begin{aligned} \sum_{\substack{k,l=3 \\ k \neq l}}^{n+1} \{\{\Psi_{kl}, \Psi_{1k}\}, \Psi_{2l}\} &= \sum_{\substack{k,l=3 \\ k \neq l}}^{n+1} (2\{\Psi_{1k}, \Psi_{2l}\}\Psi_{kl} + \{[\Psi_{kl}, \Psi_{1k}], \Psi_{2l}\} + \{[\Psi_{kl}, \Psi_{2l}], \Psi_{1k}\} \\ &\quad + [\Psi_{1k}, [\Psi_{kl}, \Psi_{2l}]]) \equiv \sum_{\substack{k,l=3 \\ k \neq l}}^{n+1} \left(-\frac{1}{2}\{\Psi_{1l}, \Psi_{2l}\} + \frac{1}{2}\{\Psi_{2k}, \Psi_{1k}\} + \frac{1}{2}[\Psi_{1k}, \Psi_{2k}] \right) \text{mod} \\ (U(\mathfrak{g})\mathfrak{k}_0)^{K_0} &= -\frac{1}{4} \sum_{\substack{k,l=3 \\ k \neq l}}^{n+1} \Psi_{12} = -\frac{(n-1)(n-2)}{4} \Psi_{12} \\ &= \frac{(n-1)(n-2)}{8} D_0. \end{aligned}$$

Formula (A.2) gives

$$\begin{aligned} \sum_{k=3}^{n+1} \{\Psi_{1k}, \{\Psi_{1k}, \Psi_{12}\}\} &= \sum_{k=3}^{n+1} (\{\{\Psi_{1k}, \Psi_{1k}\}, \Psi_{12}\} - [\Psi_{1k}, [\Psi_{1k}, \Psi_{12}]]) = 2 \left\{ \sum_{k=3}^{n+1} \Psi_{1k}^2, \Psi_{12} \right\} \\ &\quad - \frac{1}{2} \sum_{k=3}^{n+1} [\Psi_{1k}, \Psi_{2k}] = -\frac{1}{4} \{D_0, D_1\} + \frac{1}{4} \sum_{k=3}^{n+1} \Psi_{12} = -\frac{1}{4} \{D_0, D_1\} - \frac{n-1}{8} D_0. \end{aligned}$$

Finally, from (A.5) we obtain

$$[D_1, D_3] = -\{D_0, D_1\} - \frac{n-1}{2} D_0 + \frac{(n-1)(n-2)}{2} D_0 = -\{D_0, D_1\} + \frac{(n-1)(n-3)}{2} D_0.$$

Calculations of $[D_1, D_3]$ for algebras $\text{Diff}(\mathbf{P}^n(\mathbb{H})_{\mathbb{S}})$ and $\text{Diff}(\mathbf{P}^n(\mathbb{C})_{\mathbb{S}})$ are analogous, but much longer.

Let us demonstrate calculations in the octonionic case by one example. Below indices i, j vary from 0 to 7. Let D_0, \dots, D_9 be the generators of $\text{Diff}(\mathbf{P}^2(\mathbb{C}a)_{\mathbb{S}})$, $\mathfrak{g} = \mathfrak{f}_4$, $\mathfrak{k}_0 = \mathfrak{spin}(7)$, $K_0 = \text{Spin}(7)$. Then from (A.1) and proposition 5 we obtain

$$\begin{aligned} [D_1, D_3] &= \frac{1}{2} \sum_{i,j} (\{[\{e_{\lambda,i}, e_{\lambda,j}\}, e_{\lambda,i}], f_{\lambda,j}\} + \{e_{\lambda,j}, \{e_{\lambda,i}, [e_{\lambda,i}, f_{\lambda,j}]\}\}) \\ &= \frac{1}{8} \sum_{i \neq j} \{\{x C_{2,\bar{e}_i,\bar{e}_j}, e_{\lambda,i}\}, f_{\lambda,j}\} - \frac{1}{4} \sum_i \{e_{\lambda,i}, \{e_{\lambda,i}, \Lambda\}\} - \frac{1}{4} \sum_{i \neq j} \{e_{\lambda,j}, \{e_{\lambda,i}, e_{2\lambda, e_i \bar{e}_j}\}\}. \end{aligned}$$

Formulae (A.3), (60), (47) and proposition 5 imply

$$\begin{aligned}
& \frac{1}{8} \sum_{i \neq j} \{ \{ \chi C_{2, \bar{e}_i, \bar{e}_j}, e_{\lambda, i} \}, f_{\lambda, j} \} = \frac{1}{8} \sum_{i, j} (2 \{ e_{\lambda, i}, f_{\lambda, j} \} \chi C_{2, \bar{e}_i, \bar{e}_j} + \{ \{ \chi C_{2, \bar{e}_i, \bar{e}_j}, e_{\lambda, i} \}, f_{\lambda, j} \} \\
& \quad + (\{ \{ \chi C_{2, \bar{e}_i, \bar{e}_j}, f_{\lambda, j} \}, e_{\lambda, i} \} + [e_{\lambda, i}, [\chi C_{2, \bar{e}_i, \bar{e}_j}, f_{\lambda, j}]])) \\
& \equiv \sum_{i \neq j} \left(-\frac{1}{16} \{ \text{ad } Y_2(\chi C_{2, \bar{e}_i, \bar{e}_j} |_{\mathbb{C}a_2} \bar{e}_i), f_{\lambda, j} \} + \frac{1}{16} \{ \text{ad } Y_1(\chi C_{2, \bar{e}_i, \bar{e}_j} |_{\mathbb{C}a_1} e_j), e_{\lambda, i} \} \right. \\
& \quad \left. + \frac{1}{16} [e_{\lambda, i}, \text{ad } Y_1(\chi C_{2, \bar{e}_i, \bar{e}_j} |_{\mathbb{C}a_1} e_j)] + \frac{1}{2} \{ e_{\lambda, i}, f_{\lambda, j} \} f_{2\lambda, e_i \bar{e}_j} \right) \text{mod}(U(\mathfrak{g}) \mathfrak{k}_0)^{K_0} \\
& = \sum_{i \neq j} \left(-\frac{1}{4} \{ \text{ad } Y_2(\bar{e}_j), f_{\lambda, j} \} - \frac{1}{8} \{ \text{ad } Y_1(e_i), e_{\lambda, i} \} - \frac{1}{8} [e_{\lambda, i}, \text{ad } Y_1(e_i)] \right. \\
& \quad \left. + \frac{1}{4} (\{ f_{2\lambda, e_i \bar{e}_j}, \{ e_{\lambda, i}, f_{\lambda, j} \} \} - [f_{2\lambda, e_i \bar{e}_j}, \{ e_{\lambda, i}, f_{\lambda, j} \}]) \right) \\
& = \sum_{i \neq j} \left(\frac{1}{2} \{ e_{\lambda, j}, f_{\lambda, j} \} - \frac{1}{4} \{ f_{\lambda, i}, e_{\lambda, i} \} - \frac{1}{4} [e_{\lambda, i}, f_{\lambda, i}] \right) + D_8 \\
& \quad - \frac{1}{4} \sum_{i \neq j} (\{ \{ f_{2\lambda, e_i \bar{e}_j}, e_{\lambda, i} \}, f_{\lambda, j} \} + \{ e_{\lambda, i}, [f_{2\lambda, e_i \bar{e}_j}, f_{\lambda, j}] \}) \\
& = D_8 + \sum_{i \neq j} \left(\frac{1}{4} \{ e_{\lambda, j}, f_{\lambda, j} \} + \frac{1}{8} \Lambda + \frac{1}{8} \{ e_{\lambda, e_i \bar{e}_j e_i}, f_{\lambda, j} \} - \frac{1}{8} \{ e_{\lambda, i}, f_{\lambda, e_i \bar{e}_j e_j} \} \right) \\
& = D_8 + \sum_{i \neq j} \left(\frac{1}{4} \{ e_{\lambda, j}, f_{\lambda, j} \} - \frac{1}{8} \{ e_{\lambda, e_j \bar{e}_i e_i}, f_{\lambda, j} \} - \frac{1}{8} \{ e_{\lambda, i}, f_{\lambda, i} \} \right) + 7\Lambda = D_8 + 7D_0.
\end{aligned}$$

Similarly, from (A.2) and proposition 5 we get

$$\begin{aligned}
-\frac{1}{4} \sum_i \{ \{ \Lambda, e_{\lambda, i} \}, e_{\lambda, i} \} &= -\frac{1}{4} \sum_i (\{ \Lambda, \{ e_{\lambda, i}, e_{\lambda, i} \} \} + [e_{\lambda, i}, [\Lambda, e_{\lambda, i}]]) \\
&= -\frac{1}{2} \left\{ \Lambda, \sum_i e_{\lambda, i}^2 \right\} + \frac{1}{8} \sum_i [e_{\lambda, i}, f_{\lambda, i}] = -\frac{1}{2} \{ D_0, D_1 \} - \frac{1}{2} D_0.
\end{aligned}$$

Also

$$\begin{aligned}
-\frac{1}{4} \sum_{i \neq j} \{ \{ e_{2\lambda, e_i \bar{e}_j}, e_{\lambda, i} \}, e_{\lambda, j} \} &= -\frac{1}{4} \sum_{i \neq j} (\{ e_{2\lambda, e_i \bar{e}_j}, \{ e_{\lambda, i}, e_{\lambda, j} \} \} + [e_{\lambda, i}, [e_{2\lambda, e_i \bar{e}_j}, e_{\lambda, j}]]) \\
&= -\frac{1}{8} \sum_{i \neq j} [e_{\lambda, i}, f_{\lambda, e_i \bar{e}_j e_j}] = -\frac{1}{8} \sum_{i \neq j} [e_{\lambda, i}, f_{\lambda, i}] = \frac{7 \times 8}{2 \times 8} \Lambda = \frac{7}{2} D_0
\end{aligned}$$

since $e_{2\lambda, e_i \bar{e}_j}$ is antisymmetric and $\{ e_{\lambda, i}, e_{\lambda, j} \}$ is symmetric w.r.t. i, j .

Thus

$$[D_1, D_3] = D_8 + \left(7 - \frac{1}{2} + \frac{7}{2}\right) D_0 - \frac{1}{2} \{ D_0, D_1 \} = D_8 - \frac{1}{2} \{ D_0, D_1 \} + 10D_0.$$

Appendix B.

In this appendix we will prove the following theorem:

Theorem 3. *Let M be a two-point G -homogeneous Riemannian space, where G is the identity component of the isometry group for M . For every smooth vector field v on M define a function f_v on $M_{\mathbb{S}}$ by the following formula:*

$$f_v(y) = \hat{g}(v(x), \xi) \equiv \langle v(x), \xi \rangle$$

where $x \in M$, $\hat{g}(\cdot, \cdot) \equiv \langle \cdot, \cdot \rangle$ is the Riemannian metric on M , $\xi \in T_x M$, $\langle \xi, \xi \rangle = 1$, $y = (x, \xi) \in M_{\mathbb{S}}$. Let $D_0 \in \text{Diff}(M_{\mathbb{S}})$ be the differential operator constructed in section 4 (for the noncompact case, see remark 1). For every element $X \in \mathfrak{g}$ we denote by \tilde{X} the corresponding Killing vector field on M . Then the condition $D_0 f_v \equiv 0$ on $M_{\mathbb{S}}$ is equivalent to the equality $v = \tilde{X}$ for some $X \in \mathfrak{g}$. In other words, the kernel of the operator D_0 consists of functions $f_{\tilde{X}}$, where X runs over the algebra \mathfrak{g} .

This theorem for the case $M = \mathbf{H}^n(\mathbb{R})$ was formulated and proved in [4] by the explicit coordinate calculations. Here we will prove it in the general case in a more conceptual way.

Proof. Let K be the stationary subgroup corresponding to the point $x_0 \in M$, $e_0 = \frac{1}{R} \tilde{\Lambda}(x_0) \in T_{x_0} M$, $\langle e_0, e_0 \rangle = 1$, where Λ and R are from proposition 1. The space $M_{\mathbb{S}}$ is the G -orbit $G y_0$, where $y_0 = (x_0, e_0) \in M_{\mathbb{S}}$.

The action of D_0 on f_v can be written in the following way [3] (theorem 4.3):

$$(D_0 f_v)(g y_0) = \left. \frac{d}{dt} \right|_{t=0} f_v(g \exp(t \Lambda) y_0) \quad g \in G.$$

Therefore,

$$\begin{aligned} (D_0 f_v)(g y_0) &= \left. \frac{d}{dt} \right|_{t=0} \langle v(g \exp(t \Lambda) x_0), g \exp(t \Lambda) e_0 \rangle \\ &= \left. \frac{d}{dt} \right|_{t=0} \left\langle v(g \exp(t \Lambda) g^{-1} g x_0), g \exp(t \Lambda) \left. \frac{d}{d\mu} \right|_{\mu=0} \exp(\mu \Lambda) x_0 \right\rangle \\ &= \left. \frac{d}{dt} \right|_{t=0} \left\langle v(\exp(t \text{Ad}_g \Lambda) g x_0), \left. \frac{d}{d\mu} \right|_{\mu=0} \exp(t \text{Ad}_g \Lambda) \exp(\mu \text{Ad}_g \Lambda) g x_0 \right\rangle \\ &= \left. \frac{d}{dt} \right|_{t=0} \left\langle v(\exp(t \text{Ad}_g \Lambda) g x_0), \left. \frac{d}{d\mu} \right|_{\mu=0} \exp(\mu \text{Ad}_g \Lambda) \exp(t \text{Ad}_g \Lambda) g x_0 \right\rangle. \end{aligned}$$

Due to the transitivity of G -action on $M_{\mathbb{S}}$ the point $y := (x, e) := (g x_0, \widetilde{\text{Ad}_g \Lambda}|_{g x_0})$ can be considered as arbitrary. Denote $W = \text{Ad}_g \Lambda$. Then

$$(D_0 f_v)(y) = \left. \frac{d}{dt} \right|_{t=0} \langle v(\exp(t W) x), \tilde{W}(\exp(t W) x) \rangle = \mathfrak{L}_{\tilde{W}} \hat{g}(v(x), \tilde{W}(x))$$

where \mathfrak{L}_X is the Lie derivative along the vector field X . The vector field \tilde{W} is Killing, so $\mathfrak{L}_{\tilde{W}} \hat{g} = 0$ and

$$\begin{aligned} D_0 f_v &= \hat{g}(\mathfrak{L}_{\tilde{W}} v, \tilde{W}) + \hat{g}(v, \mathfrak{L}_{\tilde{W}} \tilde{W}) = \hat{g}(\mathfrak{L}_{\tilde{W}} v, \tilde{W}) = -\hat{g}(\mathfrak{L}_v \tilde{W}, \tilde{W}) \\ &= \frac{1}{2}(\mathfrak{L}_v \hat{g})(\tilde{W}, \tilde{W}) - \frac{1}{2}\mathfrak{L}_v(\hat{g}(\tilde{W}, \tilde{W})) = \frac{1}{2}(\mathfrak{L}_v \hat{g})(\tilde{W}, \tilde{W}) \end{aligned} \quad (\text{B.1})$$

due to $\hat{g}(\tilde{W}(x), \tilde{W}(x)) = \hat{g}(g \tilde{\Lambda}(x_0), g \tilde{\Lambda}(x_0)) = \hat{g}(\tilde{\Lambda}(x_0), \tilde{\Lambda}(x_0)) = R^2$ and $\mathfrak{L}_X Y = [X, Y]_c$ where $[X, Y]_c$ is the commutator of vector fields X and Y . The element $\tilde{W} \in T_x M$ is arbitrary, therefore from (B.1) we see that the condition $D_0 f_v = 0$ is equivalent to the equality $\mathfrak{L}_v \hat{g} = 0$, which means that v is a Killing vector field and has the form $v = \tilde{X}$ for some $X \in \mathfrak{g}$ if and only if $D_0 f_v \equiv 0$. \square

References

- [1] Helgason S 1973 The surjectivity of invariant differential operators on symmetric spaces *Ann. Math.* **98** 451–80
- [2] Vinberg E B 2001 Commutative homogeneous spaces and co-isotropic symplectic actions *Russ. Math. Surv.* **56** 1–60
- [3] Helgason S 1984 *Groups and Geometric Analysis* (Orlando, FL: Academic)
- [4] Reimann H M 1982 Invariant differential operators in hyperbolic space *Comment. Math. Helvetici* **57** 412–44
- [5] Shchepetilov A V 2002 Two-body problem on two-point homogeneous spaces, invariant differential operators and the mass centre concept *Preprint math-ph/0203050 (J. Geom. Phys.* at press)
- [6] Kobayashi S and Nomizu K 1969 *Foundations of Differential Geometry* vol 2 (New York: Interscience)
- [7] Tits J 1955 Sur certains classes d'espace homogenes de groupes de Lie *Acad. R. Belg. Cl. Sci. Mem. Coll.* at press
- [8] Wang H C 1952 Two-point homogeneous spaces *Ann. Math.* **55** 177–91
- [9] Matsumoto H 1971 Quelques remarques sur les espaces riemanniens isotropes *C. R. Acad. Sci., Paris.* **272** 316–9
- [10] Wolf J A 1972 *Spaces of Constant Curvature* (Berkeley, CA: University of California Press)
- [11] Chavel I 1972 *Riemannian Symmetric Spaces of Rank One* (New York: Dekker)
- [12] Helgason S 1978 *Differential Geometry, Lie Groups, and Symmetric Spaces* (New York: Academic)
- [13] Loos O O 1969 *Symmetric Spaces. V. II: Compact Spaces and Classification* (New York: Benjamin)
- [14] Collingwood D H 1985 *Representations of Rank One Lie Groups (Research Notes in Mathematics vol 137)* (Boston, MA: Pitman)
- [15] Stepanova I E and Shchepetilov A V 2000 Two-body problem on spaces of constant curvature: II. Spectral properties of the Hamiltonian *Theor. Math. Phys.* **124** 1265–72
- [16] Baez J C 2001 The octonions *Bull. Am. Math. Soc.* **39** 145–205
- [17] Postnikov M M 1982 *Lie Groups and Lie Algebras* (Moscow: Nauka) (in Russian)
- [18] Adams J F 1996 *Lectures on Exceptional Lie Groups* (Chicago, IL: University of Chicago Press)
- [19] Onishchik A L 1994 *Topology of Transitive Transformation Groups* (Leipzig: Barth)